Spectrum-Based exponential stability analysis of linear time-invariant fractional delay differential algebraic systems

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Outline

We are interested in the delay-independent stability of system of equations of Fractional Delay Differential-Algebraic Equation (FDDAE)

* Some definitions and facts used in this talk;

* Stability Criteria: Sufficient conditions for delay-independently asymptotic stability of FDDAE are derived to ensure exponential stability for any delay parameter based on

Spectrum-based stability analysis

Delay–independent asymptotic stability

Introduction and previous Works

Consider the delay differential Algebraic system of Caputo fractional-order :

$$ED^{\alpha}y(t) = Ay(t) + By(t-\tau) + f(t) \qquad t \geq 0 \qquad (1)$$

$$y(t) = \varphi(t) \qquad -\tau \leq t \leq 0$$

where $0 \leq \alpha \leq 1$; $\mathrm{y(t)} \in \mathrm{R^n}$ is the state vector. $D^{\alpha}y(t)$ denotes an α order Caputo fractional –order derivative of y(t) ; A, B, E ∈ $R^{n\times n}$ with E singular and rank(E)=r < n ; $\tau \in R^{+}$ is the time delay and φ is a consistent initial function.

Introduction and previous Works

The analysis of stability of integer-order delay differential algebraic systems have been considered by many researchers:

Ha, 2015 –analysis and numerical solutions of delay differential –algebraic equations

Yuan and Shen, 2014 - The stability of two-step Runge-Kutta methods for Neutral Delay integro differential – algebraic equations with many delays

Hu,Cong and Hu, 2018 -Delay–dependent stability of linear multistep methods for DAEs with Multiple delays

Milano, 2016- Small-signal stability analysis for non-index 1 Hessenberg form systems of delay differential-algebraic equations

Ha, 2018- Spectral characterizations of solvability and stability for delay differential-algebraic equations

Ha, 2018- On the stability analysis of delay differential-algebraic equations

Liu, Dassios and Milano, 2018-On the stability analysis of systems of Neutral delay differential equations.

Introduction and Previous Works

The analysis of stability of fractional-order systems is more complex than that of integer-order differential systems. In recent years, some results have been obtained on the stability analysis for fractional-order delay differential equations. For example:

Zhang,Wu and Cao 2014 –asymptotic stability of caputo type fractional neutral dynamical systems

Li, Zhong and Li, 2015 – asymptotic stability analysis of fractional-order neutral systems with time delay

Priyadharsini and Govindaraj, 2018 –asymptotic stability of caputo fractional singular differential systems with multiple delays

Zaczkiewicz, 2019 Fractional differential algebraic systems with delay : computation of final dimension initial conditions and inputs for given outputs

Cermak and Kisela , 2019- oscillatory and asymptotic properties of fractional delay differential equations

Preliminaries

Throughout this talk, for any matrix X,

det(X) represents the determinant of X.
$\sigma(X)$ denotes the spectrum of X,
$\rho(X)$ represents the spectral radius of X and
$arg(\sigma(X))$ stands for the principle argument of $\sigma(X)$.
$C^{-} = \{\lambda \in C; Re(\lambda) < 0\}$

Preliminaries

Definition 1 [Podlubny,1999] Riemann- Liouville's fractional integral of order q>0 for a function $f: R^+ \rightarrow R^+$ is defined as:

$$D^{-q} f(t) = \frac{1}{\Gamma(q)} \int_0^b (t-s)^{q-1} f(s) ds \qquad t > 0$$

where Γ(.) is the gamma function

Definition 2 [Podlubny,1999] The Caputo's fractional derivative of order q for a function $f: R^+ \rightarrow R^+$ is defined as:

$$D^{q} f(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-q-1} f(s) ds$$

$$0 \leq m-1 < q < m, \quad m \in Z^+$$

Preliminaries

Definition 3

Consider the FDDAE (1). The matrix triple (E, A, B) is called regular if for any given matrices E, A, B $\in \boldsymbol{R}^{n,n}$ **:**

$\mathrm{Q(s,\tau)} = \det(s^{\alpha}E - A - Be^{-s\tau})$ **, which is called the characteristic polynomial of system (1), is not identically zero, where** $s \in C$.

Equivalently, the matrix triple (E, A, B) is called regular if and only if the spectrum

$$\sigma(E,A,B) := \{\lambda \in C \,|\, \det(s^{\alpha}E - A - Be^{-s\tau}) = \mathbf{0}\}$$

is not the entire C. Elements of the spectrum $\sigma(E,A,B)$ **are called eigenvalues of the triple (E, A, B).**

Preliminaries

Lemma 1 [Ha, 2018]

Assume that an initial function $x|[-\tau, 0]$ is consistent and sufficiently smooth. Then the FDDAEs (1) is uniquely solvable if and only if the matrix triple (E, A, B) is regular.

Definition 4 [Podlubny,1999]

Assume that (E,A,B) is regular. Then the zero solution y(t)=0 of system (1) is called delay-independent asymptotically stable (exponential stability) if for any consistent $\varphi(.) \in C([-\tau, 0], R^n)$ its analytic solution y(t) satisfies $\lim_{t \to +\infty} y(t) = 0$ for any delay parameter $\tau > 0$.

Stability Criteria

We derive the sufficient conditions for the delay- independently exponential stability of system (1).

Spectrum-based stability analysis

Lemma 2

If all roots of the characteristic equation

$$Q(s,\tau) = det[s^{\alpha}E - A - Be^{-s\tau})] = 0$$

lie in the open left-half complex plane and are uniformly bounded away from the imaginary axis, then the zero solution of system (1) is delay-independent globally asymptotically stable.

Spectrum-based stability analysis

Lemma 3

Suppose that :

$$(C_1) \quad |\arg(\sigma(E, A, B))| > \frac{\alpha\pi}{2}$$

$$(C_2) \quad \sup_{\mathrm{Re}(s)\geq 0} \rho((s^{\alpha}E - A - B)^{-1}B) < \frac{1}{2}$$

Then :

$(\widetilde{C_2})\ Q(s,\tau) = det[s^{\alpha}E - (A + Be^{-s\tau})] \neq 0 \quad for\, all\ \ s \in C$ **such that Re(s)=0.**

Stability Criteria

Spectrum-based stability analysis

Theorem 1

Suppose that the triple (E, A, B) is regular and that :

$$(C_1)\quad |\arg(\sigma(E, A, B))| > \frac{\alpha\pi}{2}$$

$$(C_2)\quad \sup_{Re(s)\geq 0} \rho((s^{\alpha}E - A - B)^{-1} B) < \frac{1}{2}$$

Or $(\widetilde{C_2})\ Q(s,\tau) = det[s^{\alpha}E - (A + Be^{-s\tau})] \neq 0 \quad for\ all\ s \in C$ **such that Re(s)=0.**

Then the DDAE(1) is exponentially stable for all values of the delay τ, i.e. the exponential stability of (1) is delay – independent.

Spectrum-based stability analysis

Theorem 2

Assume that the triple (E,A,B) is regular, then system (1) is delay-independent exponentially stable if:

$$(C_1)\quad |\arg(\sigma(E,A,B))| > \frac{\alpha\pi}{2}$$

$$(C_2)\quad \sup_{Re(s)\geq 0} \rho((s^{\alpha}E - A - B)^{-1}B) < \frac{1}{2}$$

or $(\widetilde{S_2})$ **for any** $s \in \mathrm{C},\ Re(s) = 0\ we\ have$

$$\sigma((A + B - s^{\alpha}E)^{-1}B - \frac{1}{2}I) \in C^{-}$$

Spectrum-based stability analysis

An Illustrative Example:

Consider the Caputo fractional-order delay differential algebraic equation:

$$D^{1/3}y_1(t) = -y_1(t) + y_2(t-1) \qquad (2)$$
$$0 = -y_2(t).$$

Then we have $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

We have $\sigma(E, A, B) \in C^{-}$ **and**

$$\left(A + B - s^{1/3}E\right)^{-1}B - \frac{1}{2}I = \begin{bmatrix} 0 & \dfrac{1}{-1-s^{1/3}} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \dfrac{1}{2} & 0 \\ 0 & \dfrac{1}{2} \end{bmatrix}$$

Stability Criteria

Spectrum-based stability analysis

$$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{-1-s^{1/3}} \\ 0 & -\frac{1}{2} \end{bmatrix}$$

From the equation $det[\lambda E - A - B] = 0$ **we have** $|\arg(\lambda)| = \pi > \frac{\alpha\pi}{2}$.

Moreover, $det\left[\lambda E - \left(\left(A + B - s^{1/3}E\right)^{-1} B - \frac{1}{2} I\right)\right] = 0 \Rightarrow (\lambda + \frac{1}{2})^2 = 0$

So that $Re\left(\left(\left(A + B - s^{1/3}E\right)^{-1} B - \frac{1}{2} I\right)\right) < 0$. **Thus conditions** (C_1) **and** $(\widetilde{S_2})$ **are satisfied.**

By theorem 2, the system (2) is delay independent asymptotically stable.

Stability Criteria

Delay – independent asymptotic stability

For a complex matrix W, let $\mu(W)$ be the logarithmic norm of W:

$$\mu(W) = \lim_{\Delta \to 0^+} \frac{\|I + \Delta W\| - 1}{\Delta} \ .$$

$\mu(W)$ depends on the chosen matrix norm. Let $\|W\|$ denote the matrix norm of W subordinate to a certain vector norm. In order to specify the norm, the notation $\|.\|_\rho$ is used and the notation $\mu_\rho(.)$ is also adopted to denote the logarithmic norm associated with $\|.\|_\rho$.

The following theorem is a sufficient condition for the stability of (1). Here, the matrix A is required to be nonsingular.

Stability Criteria

Delay – independent asymptotic stability

Theorem 3

Let $\|A^{-1}\|.\|B\| < 1.$ Suppose that the triple (E, A, B) is regular and that :

$$(C_1) \quad |\arg(\sigma(E,A,B))| > \frac{\alpha\pi}{2}$$

$(\widetilde{C_2})$ $Q(s,\tau) = det[s^{\alpha}E - (A + Be^{-s\tau})] \neq 0 \quad for\ all\ \ s \in C$ **such that Re(s)=0.**

$or\ (C_3)$ **the condition** $\mu(EA^{-1}) + \frac{\|E\|}{\frac{1}{\|A^{-1}\|} - \|B\|} < 0$ **holds;**

Then system (1) is asymptotically stable.

Conclusions

- **Much less work is done on the stability analysis of fractional-order delay differential algebraic systems.**

- **For such systems, we proposed some sufficient conditions to ensure the asymptotic stability :**
  - **a spectrum based approach, illustrated with an example,**
  - **a logarithmic norm based approach.**

Thank you