Preservation of log concavity for positive linear operators and applications in deterioration models

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1. Introduction

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Log concave functions

Definition 1 *Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to [0, \infty]$ is said to be log concave on I if for all $x, y \in I$ and $0 \leq \alpha \leq 1$ it verifies that*

$$f(\alpha x + (1-\alpha)y) \geq f(x)^{\alpha} f(y)^{1-\alpha}, \tag{1}$$

or equivalently $\log f$ is concave (in the interval in which f is strictly positive).

Remark 1 *If the inequality in the previous Definition is reversed, we obtain the dual concept of log convexity.*

Some applications

- Reliability theory: When studying lifetimes, log concavity of certain probabilities indicates the adverse effect of the age in the system.
- Operations Research: Log concave functions are unimodal (in fact, strongly unimodal). Log concave cost functions ensure the existence of an optimal value.

Positive linear operators

Consider an interval J, and a positive linear operator T of the form:

$$Tf(t) = \int_I f(u)d\mu_t(u), \quad t \in J, \quad f \in \mathcal{T}, \tag{2}$$

- $\mathcal{T}$ is the set of measurable functions $f : I \to \mathbb{R}$ which are μ_t-integrable for all $t \in J$
- $\{\mu_t,\ t \in J\}$ is a family of probability measures concentrated on I
- That is, if $\{X(t),\ t \in J\}$ are random variables such that $X(t)$ has distribution μ_t, then we can write

$$Tf(t) = Ef(X(t)), \quad t \in J.$$

Our aim: To give conditions on T, such that

$$f \text{ log concave } \Rightarrow Tf \text{ log concave .}$$

- From now on $J = [0, \infty)$: $(X(t), \quad t \geq 0)$ stochastic process
- $I = [0, \infty)$ or $(0, \infty)$: ($X(t)$ non-negative random variables).

Examples: $f : [0, \infty) \to \mathbb{R}$

- **Szász operator**: $Lf(t) = Ef(N(t)) = \sum_{k=0}^{\infty} f(k) e^{-t} \frac{t^k}{k!}$,

  $(N(t),\ t \geq 0)$: Each $N(t)$ has Poisson distribution with mean t

$$P(N(t) = k) = e^{-t} \frac{t^k}{k!}, k = 0, 1, 2, \ldots$$

- **Gamma operator**: $Gf(t) = Ef(S(t)) = \int_0^{\infty} f(x) \frac{x^{t-1}}{\Gamma(t)} e^{-x} dx$,

  $(S(t),\ t \geq 0)$: Each $S(t)$ has Gamma distribution with shape parameter t, i.e. density function

$$f_t(x) = \frac{x^{t-1}}{\Gamma(t)} e^{-x}, \quad x > 0$$

- **Szász-Durrmeyer operator**: $M^* f(t) = Ef\left(\sum_{i=1}^{N(t)} X_i \right)$.

  $X_1, X_2, \ldots$: sequence of independent identically distributed (i.i.d.) non negative random variables.

Previous work

- Preservation properties in positive linear operators:
  - Adell, J.A. y Perez-Palomares, A. [8] (1999): Use of stochastic orders in preservation properties
  - Adell, J.A. y Lekuona, A. [7] (2001): Preservation of concavity and convexity
  - Badía, F.G. (2009) [13]: Preservation of log convexity (f log convex $\Rightarrow f$ convex).
- Our aim: To study the preservation of log-concavity:
  - Tool: stochastic orders.
  - f concave $\Rightarrow f$ log-concave.
  - Log-concave functions not being concave: $f(x) = 1_J(x)$, for an interval J, or $f(x) = x^a$ whenever $a > 1$
  - The techniques in [13] for the preservation of log-convexity cannot be extended to log-concavity.

Stochastic orders

Let X_1, X_2 be two random variables defined in the same probability space $(\Omega, \mathcal{F}, P)$

$$\begin{aligned} X_i : \ \Omega \ &\to \mathbf{R} \\ \omega \ &\rightsquigarrow X_i(\omega), \quad i = 1, 2. \end{aligned}$$

- "Natural stochastic order" $X_1 \leq_{a.s.} X_2$ iff $X_1(\omega) \leq X_2(\omega)$ P-almost surely.
- The previous stochastic order is not the usual one.
- The most common stochastic orders compare
  - The distribution functions: $F_{X_i}(x) := P(X_i \leq x)$
  - The survival functions: $\bar{F}_{X_i}(x) := P(X_i > x)$
  - The densities f_{X_i} (if X_1 and X_2 are absolutely continuous with respect to a common real measure μ, that is:

$$P(X_i \in A) = \int_A f_{X_i}(x) d\mu(x), \quad \text{for Borel sets } A, \quad i = 1, 2$$

Stochastic orders: definitions

Definition 2 *Let X_1 and X_2 be two random variables. X_1 is said to be smaller than X_2 in*

(i) The usual stochastic order (written as $X_1 \leq_{st} X_2$) if $\bar{F}_{X_1}(x) \leq \bar{F}_{X_2}(x)$, for all $x \in \mathbb{R}$;

(ii) The hazard rate order ($X_1 \leq_{hr} X_2$) if $\overline{F}_{X_2}(x)/\overline{F}_{X_1}(x)$ is increasing in x;

(iii) The reversed hazard rate order ($X_1 \leq_{rh} X_2$) if $F_{X_2}(x)/F_{X_1}(x)$ is increasing in x;

(iv) The likelihood ratio order ($X_1 \leq_{lr} X_2$) if X_1 and X_2 are absolutely continuous with respect to some dominating measure μ, with respective densities f_{X_1} and f_{X_2} such that $f_{X_2}(x)/f_{X_1}(x)$ is increasing in x.

Remark 2
- $X_1 \leq_{lr} X_2 \Rightarrow X_1 \leq_{hr} X_2$ *and* $X_1 \leq_{rh} X_2$;
- *Either* $X_1 \leq_{hr} X_2$ *or* $X_1 \leq_{rh} X_2 \Rightarrow X_1 \leq_{st} X_2$.

Stochastic orders: motivation

- $X_1 \leq_{st} X_2$ guarantees that

$$P(X_1 > x) \leq P(X_2 > x), \quad x \in \mathbf{R}$$

- $X_1 \leq_{hr} X_2$ guarantees that

$$P(X_1 > x + h | X_1 > x) \leq P(X_2 > x + h | X_2 > x), \quad h > 0$$

- $X_1 \leq_{lr} X_2$ guarantees that

$$P(X_1 > x + h | X_1 \in A) \leq P(X_2 > x + h | X_2 \in A),$$

  for each Borel set A with $(x, \infty) \subseteq A$.

Books on stochastic orders: Müller and Stoyan (2001)[22], Shaked and Shanthikumar (2007)[31]

Example Gamma operator

$$Gf(t) = Ef(S(t)) = \int_0^\infty f(x) \frac{x^{t-1}}{\Gamma(t)} e^{-x} dx,$$

We have that the random variables in $(S(t),\ t \geq 0)$ are ordered in the likelihood ratio.

That is, for $0 < t_1 < t_2$, $S(t_1) \leq_{lr} S(t_2)$

In fact,

$$\frac{f_{t_2}(x)}{f_{t_1}(x)} = \frac{\dfrac{x^{t_2-1}}{\Gamma(t_2)} e^{-x}}{\dfrac{x^{t_1-1}}{\Gamma(t_1)} e^{-x}} = \frac{\Gamma(t_1)}{\Gamma(t_2)} x^{t_2-t_1}$$

which is increasing in x if $0 < t_1 < t_2$.

Moreover, $S(t_1) \leq_{lr} S(t_2)$ implies $S(t_1) \leq_{hr} S(t_2)$ and $S(t_1) \leq_{rh} S(t_2)$.

Positive linear operators in terms of stocastic processes

- Our aim is to study the log concavity of

$$Tf(t) = \int_I f(u)\,d\mu_t(u) = Ef(X(t)), \quad t \geq 0$$

- To this end, we will consider a stochastic process $(X(t),\ t \geq 0)$.
- That is we will consider that the random variables are defined in a common probability space $(\Omega, \mathcal{F}, P)$
- in such a way that

$$\begin{aligned} X(t):\ \Omega &\to \mathbf{R} \\ \omega &\rightsquigarrow X(t,\omega), \qquad t \geq 0. \end{aligned}$$

Tools in the study:

- Stochastic orders: we only compare the 'marginal' distribution of $X(t)$
- Dependence structure between the random variables in the process.

Classes of stochastic processes in our results

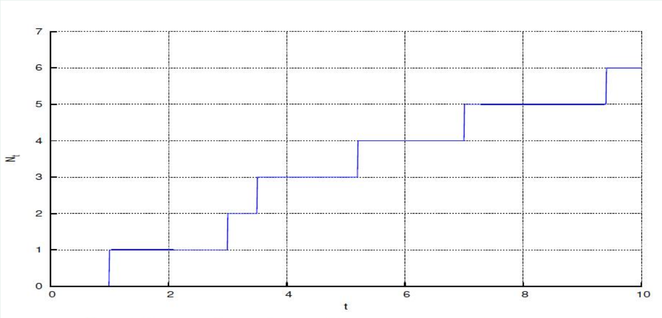
Definition 3 *Let* $(X(t),\ t \geq 0)$ *be a stochastic process. We will say that this process belongs to the class* $\mathcal{IPII}$ *(independent positive increasing increments) if it verifies:*

1. $0 \leq X(s,\omega) \leq X(t,\omega)$ *a.s., for* $0 \leq s < t$;
2. *The process has independent increments, that is: for given* $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$,
$$(X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1}))$$
is a vector of independent random variables
3. *The increments of the process satisfy:*
$$X(t+h) - X(t) \uparrow_{st} \quad \text{in } t \quad \text{for any fixed } h > 0;$$

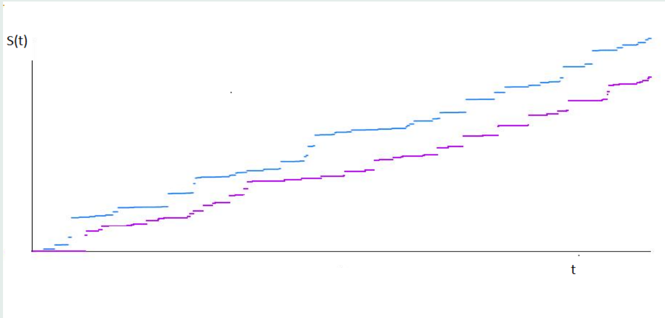
Remark 3
- *If Condition 3 is replaced by* $X(t+h)-X(t) \downarrow_{st}$ *in* t*, we will say that the process belongs to the class* $\mathcal{IPDI}$ *(independent positive decreasing increments)*
- *if the process belongs to* $\mathcal{IPDI} \bigcap \mathcal{IPII}$*, we will say that the process belongs to the* $\mathcal{IPSI}$ *class (independent positive stationary increments).*

- *We can define* *Szàsz*, *Gamma* *and* *Szàsz-Durrmeyer* *operators in terms of stochastic processes belonging to the* $\mathcal{IPSI}$ *class (Lévy subordinators, see Sato* **[5]***)*

- *Sample path of a Poisson process (Szàzs operator)*



- *Sample paths of a Gamma process (Gamma operator)*



2. Main results

Theorem 1 *Let $(X(t),\ t \geq 0)$ be a stochastic process of non-negative random variables. Consider the positive linear operator*

$$Tf(t) = Ef(X(t)), \quad t \geq 0,$$

defined on the class $\mathcal{T}$ of measurable functions $f : [0,\infty) \to \mathbf{R}$ such that $E|f|(X(t)) < \infty, t \geq 0$. Then, we have:

(a) Assume that f is log concave and decreasing.
If $(X(t),\ t \geq 0)$ is in the $\mathcal{IPII}$ class and $X(t) \uparrow_{rh}$, then Tf is a log concave and decreasing function on $[0,\infty)$.

(b) Assume that f is log concave and increasing.
If $(X(t),\ t \geq 0)$ is in the $\mathcal{IPDI}$ class and $X(t) \uparrow_{hr}$, then Tf is a log concave and increasing function on $[0,\infty)$.

(c) Assume that f is log concave.
If $(X(t),\ t \geq 0)$ is in the $\mathcal{IPSI}$ class and $X(t) \uparrow_{lr}$, then Tf is a log concave function on $[0,\infty)$.

Application to specific operators

Theorem 2 *(deduced from Theorem 1) Let* $f : [0,\infty) \longrightarrow [0,\infty)$ *be a log concave function. Then, we have*

(a) For the **Szász operator***:*

$$Lf(t) = Ef(N(t)) = \sum_{k=0}^{\infty} f(k) e^{-t} \frac{t^k}{k!},$$

Lf *is log concave on* $[0,\infty)$ *if* $f \in \mathcal{T}$

(b) For the **Gamma operator**

$$Gf(t) = Ef(S(t)) = \int_0^{\infty} f(x) \frac{x^{t-1}}{\Gamma(t)} e^{-x} dx,$$

$Gf(t)$ *is log concave on* $[0,\infty)$ *if* $f \in \mathcal{T}$

Theorem 2 (continuation)

(c) For the **Szász-Durrmeyer operator** $M^*f(t)$

$$M^*f(t) = Ef\left(\sum_{i=1}^{N(t)} X_i\right),$$

M^*f *is log concave on* $[0,\infty)$
if X_1 *is an absolutely continuous random variable, having log concave density, and* $f \in \mathcal{T}$.

(d) $M^*f(t)$ *is a log concave decreasing function on* $[0,\infty)$
if F_{X_1} *is log-concave*
and $f \in \mathcal{T}$ *is decreasing.*

(e) $M^*f(t)$ *is a log concave increasing functions on* $[0,\infty)$
if $\overline{F}_{X_1}$ *is log-concave and*
$f \in \mathcal{T}$ *is increasing.*

Applications to the incomplete gamma function

Corollary 1 *(deduced from Theorem 2 (b)) Let* $J \subseteq [0,\infty)$ *be an interval.*

Then, the function

$$f_J(t) := \int_J \frac{u^{t-1}}{\Gamma(t)} e^{-u} du, \quad t > 0,$$

is a log concave function.

Remark 4 *This corollary gives a probabilistic proof of the log concavity in the shape parameter* t *for*

- *The lower normalized gamma incomplete function (*$J = [0,x]$*)*
- *The upper normalized gamma incomplete function (*$J = (x,\infty)$*)*

Analytic proofs appear in [9] and [10], respectively.

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3. Applications to deterioration models

Deterioration models

Deterioration models aim to describe the deterioration of a system by a stochastic process $(X(t),\ t \geq 0)$

In which $X(t)$ is the degree of deterioration at instant t

Stochastic models considered:

- Gamma process: see Nicolai, Frenk and Dekker [23], Van Noortwijk [24] or Paroissin and Salami [27]
- Shock models: The deterioration is described by

$$\sum_{i=1}^{N(t)} X_i, \quad \text{in which}$$

  - $N(t)$: number of shocks in $[0, t]$
  - X_i: deterioration in each shock.

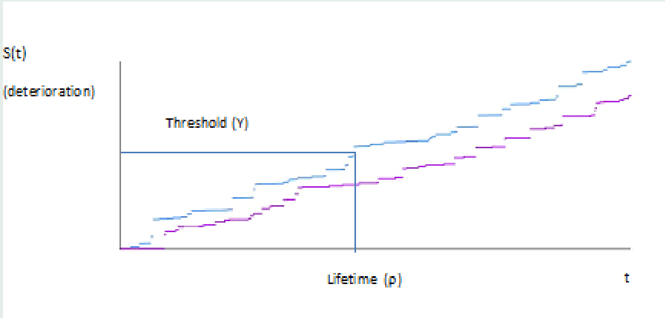
  see Abdel Hammeed and Proschan [3]

Deterioration models

In these models, the system ”breaks down” when the degree of deterioration attains a certain (random) threshold Y.

Thus, ρ, the lifetime of the system is defined by

$$\rho = \inf\{t \geq 0 : \ X(t) \geq Y\}.$$



Question: Which shape (age) properties of Y inherits ρ?.

Some previous work

Non-random threshold $Y = c$:

- Shaked and Shanthikumar [30]: Showed common ageing properties of ρ in a context of pure-jump processes
- Belzunce, Ortega, Ruiz [12]: Different ageing properties for increasing Markov processes

Random threshold Y:

- Esary, Marshall and Proschan [16]: Showed common ageing properties of ρ in shock models
- Abdel Hameed: Showed age properties for a Gamma wear process [1], and for general pure-jump processes [2, 4]. See also [5] for a review.

Age (shape) properties of lifetimes

Definition 4 *Let X be a nonnegative random variable , with F and $\overline{F} := 1 - F$ their corresponding distribution and survival functions. X (or F) is said to be*

1. *Increasing failure rate (IFR) if $\overline{F}$* is *log-concave*.
2. *Decreasing Failure Rate (DFR) if $\overline{F}$* is *log-convex*.
3. *Decreasing reversed hazard rate (DRHR) if F* is *log-concave*.

Remark 5 *Note that:*

- *The IFR is a positive ageing concept (the system gets worse with age), as*

  $\overline{F}$ *log concave* $\Rightarrow P(X > z+t|X > t)$ *is decreasing in t for fixed z.*

- *The DFR is a negative ageing concept (the system improves with age), as*

  $\overline{F}$ *log convex* $\Rightarrow P(X > z+t|X > t)$ *is increasing in t for fixed z.*

Book of life distributions in reliability: Marshall and Olkin (2007)[20]

Basic expression

Assume that $(X(t),\ t \geq 0)$ has increasing and right-continuous paths.

Y: Random Threshold

$\rho = \inf\{t \geq 0 :\ X(t) \geq Y\}$:   Lifetime of the device

We have that:

$$\bar{F}_\rho(t) := P(\rho > t) = E[\bar{F}_Y(X(t))] \tag{3}$$

and

$$F_\rho(t) := P(\rho \leq t) = E[F_Y(X(t))], \quad t \geq 0. \tag{4}$$

Non-homogeneous compound Poisson process: preservation of IFR and DRHR classes

Proposition 1 *Let $X(t)$ be a non-homogeneous compound Poisson wear process, that is*

$$X(t) = \sum_{n=1}^{N(a_2(t))} X_n,$$

in which

- $(N(t),\ t \geq 0)$ *is a Poisson process,*
- $a_2(t),\ t \geq 0$ *is an increasing function,*
- $(X_n)_{n \in \mathbb{N}^*}$ *is a sequence of i.i.d. non-negative r.v., independent of the process.*

Let ρ be the lifetime of the device. We have the following.

(a) Assume that a_2 is convex with $a_2(0) = 0$ and X_n are DRHR.

Then, Y IFR $\Rightarrow \rho$ IFR.

(b) Assume that a_2 is concave and X_n are IFR.

Then, Y DRHR $\Rightarrow \rho$ DRHR.

Non-homogeneous compound Poisson process: preservation of the IFR class

Remark 6 *Previous conditions improve those given in Abdel Hameed for the IFR property (see [2, Thm. 2.3 (i)]).*

These conditions required X_1 to have a log-concave density.

Proposition 1 (a) only requires X_1 to be DRHR (this class contains both log-concave and log-convex densities).

Non-homogeneous gamma process with trend: preservation of the IFR and DRHR property

Corollary 2 *Let $(S(t),\ t \geq 0)$ be a gamma wear process. Consider a wear process in which $(S^*(t),\ t \geq 0)$ is defined as:*

$$S^*(t) = a_1(t) + S(a_2(t)), \quad t \geq 0. \tag{5}$$

Let ρ be the lifetime of the device.

(a) Assume that a_1 and a_2 are increasing and convex, with $a_1(0) = a_2(0) = 0$.

Then, Y IFR $\Rightarrow \rho$ IFR.

(b) Assume that $a_1(t) = 0$ (no trend) and a_2 is increasing and concave.

Then, Y DRHR $\Rightarrow \rho$ DRHR.

Remark 7

- *Abdel Hammeed [1] proved the IFR property for non-homogeneus Gamma wear process, when a_2 is convex.*
- *The previous result allows us to add a convex deterministic trend.*
- *For the DRHR property, we cannot add a general deterministic trend. For instance, consider $S^*(t) = t + S(t)$. We can see that $S^*(s) \not\leq_{hr} S^*(t)$ if $0 < s < t < 1$*

Preservation of the DFR property

Proposition 2 *Let $(X(t),\ t \geq 0)$ be a process in the $\mathcal{IPSI}$ class. Consider a wear process defined as*

$$X^*(t) = a_1(t) + X(a_2(t))$$

with a_1 and a_2 being increasing and concave functions.

Let ρ be the lifetime of the device.

If Y is DFR, then ρ is DFR.

Remark 8 *Notice that.*

- *This is a result concerning preservation of log convexity.*
- *The conditions concerning the underlying process are more general, as for this result we only need the usual stochastic ordering among the random variables in the process.*
- *Observe that the previous result includes gamma or compound Poisson processes with trend.*

Preservation of log concave densities

Definition 5 *Let X be an absolutely continuous nonnegative random variable , with f its density function. X (or f) is said to be*

1. *Log-concave if f is log-concave*
2. *Log-convex if f is log-convex*

Remark 9 • f log-concave $\Rightarrow$ X *IFR* and *DRHR*

• f log-convex $\Rightarrow$ X *DFR* $\Rightarrow$ X *DRHR*

Example 1 *The gamma density* $f(x) = \dfrac{1}{\Gamma(\nu)} x^{\nu-1} e^{-x}, \ x > 0, \ \nu > 0$ *is*

- *Log-concave (IFR) if* $\nu \geq 1$
- *Log-convex (DFR) if* $0 < \nu \leq 1$
- *If* $\nu = 1$ *(exponential density) we have both properties.*

Preservation of log-concave densities

- We assume that $(X(t),\ t \geq 0)$ is a centered subordinator (it is in $\mathcal{IPSI}$ class and $E[X(t)] = t$)

- Assume that the random threshold Y is absolutely continuous, having density f_Y. We have the following representation (cf. [7, Prop. 4]) for $t > 0$:

$$f_\rho(t) = \frac{d}{dt}E[F_Y(X(t))] = E[f_Y(X(t) + UT)]$$

  where U and T are independent random variables such that

  - U is uniformly distributed on $[0, 1]$
  - T is a suitable nonnegative random variable.

- Moreover, if we consider $X^*(t) := X(a_2(t))$, in which $a_2(t)$ is an increasing and differentiable function, we have

$$f_\rho(t) = \frac{d}{dt}E[F_Y(X^*(t))] = {a_2}'(t)E[f_Y(X(a_2(t)) + UT)].$$

Preservation of log-concavity. Compound Poisson process

Proposition 3 *Let* $(X(t),\ t \geq 0)$ *be a compound Poisson process, that is*

$$X(t) = \sum_{n=1}^{N(t)} X_n,$$

in which

- $(N(t),\ t \geq 0)$ *is a homogeneous Poisson process*
- $(X_n)_{n\in\mathbb{N}^*}$ *is a sequence of i.i.d. non-negative random variables, independent of the process*

Let Y *be a log concave random threshold and let* ρ *be the lifetime of the device.*

If X_1 *is log concave, then* ρ *is log concave.*

Remark 10 *The log-concavity of* X_1 *was the condition used in [2] to prove the IFR property for* ρ*.*

Preservation of log-concavity for the gamma process

Let $(S(t),\ t \geq 0)$ be a gamma wear process. Consider

$$S^*(t) = S(a_2(t)), \quad t \geq 0.$$

We have in this case, for an increasing function a_2

$$\frac{d}{dt} E F_Y(S^*(t)) = a_2'(t) E f_Y(S(a_2(t)) + UT), \quad t > 0,$$

In which T is an exponential random variable.

We are able to show the following stochastic ordering properties:

- $S(a_2(t)) + UT \uparrow_{rh}$. This guarantees the preservation of log-concave decreasing densities
- $S(a_2(t)) + UT \uparrow_{lr}$, for $a_2(t) \geq 1$. This guarantees the preservation of log-concavity on this interval

Preservation of log-concavity for the gamma process. Decreasing log-concave density of Y

Proposition 4 *Let $(S(t),\ t \geq 0)$ be a gamma wear process. Consider a wear process in which*

$$(S^*(t) = S(a_2(t)),\ t \geq 0).$$

Assume that Y, the random threshold, has a log concave and decreasing density.

Further, asssume that $a_2(0) = 0$, a_2 is differentiable, with ${a_2}'$ is non-negative, increasing and log concave.

Then, ρ, the lifetime of the device is log concave.

Remark 11 *Examples in which conditions are verified:*

- *Random threshold Y: Exponential, or having a uniform distribution on the interval $(0, a)$, for some $a > 0$*
- *$a_2(t) = e^{ct} - 1,\ t \geq 0$ or $a_2(t) = ct,\ t \geq 0$ for $c > 0$ verifies the conditions in Proposition 4.*

Preservation of log-concavity for the gamma process. General log-concave density of Y

Proposition 5 *Let $(S(t),\ t \geq 0)$ be a gamma wear process.*

Let Y be the random threshold, and let ρ be the lifetime of the device.

If Y has a log concave density, then ρ is log concave on $[1, \infty)$.

Remark 12 *For the proof, we show that $S(t) + UT \uparrow_{lr}$, if $t \geq 1$. The proof is based on the following facts:*

- $V = S(1) + UT$ *is log-concave (this is not immediate, as UT is log-convex).*
- $S(t) \uparrow_{lr}$
- *Then, for $1 \leq t_1 < t_2$,*

  $$S(t_1) + UT =_{st} S(t_1 - 1) + V \leq_{lr} S(t_2 - 1) + V =_{st} S(t_2) + UT$$

- *Therefore, $S(t) + UT \uparrow_{lr}$, for $t \geq 1$.*
- *Question: Is $S(t) + UT \uparrow_{lr}$, for $0 \leq t < 1$?*

Future work

- Extension to the multidimensional case. That is, preservation properties for
$$Tf(t) = E[f(\mathbf{S}(t))]$$
in which
  - $(\mathbf{S}(t),\ t \geq 0)$ is a multidimensional Levy process ($\mathbf{S}(t)$ takes values on $\mathbf{R}^d$)
  - $f : I \to \mathbf{R}_+$ is a log concave function, in which $I \subseteq \mathbf{R}^d$
- Example: Deterioration in railways by a bivariate gamma process: Mercier et al. (2012) [21]
- Technique: multivariate stochastic orders (many properties are lost from the univariate orders to the multivariate order)
- Possible problem: to check the ordering conditions in t for $\mathbf{S}(t)$

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Operator not preserving log concavity

We present a operator in which conditions in Theorem 1 (c) are not satisfied, and does not preserves log concavity

Consider the Poisson process $(N(t),\ t \geq 0)$

Define the process $(t + N(t),\ t \geq 0)$, (a Poisson process with trend), and its associated operator

$$Tf(t) = Ef(t + N(t)), \qquad t \geq 0$$

Consider the log concave function $f(x) = 1_{\{2\}}(x)$. We have

$$Tf(t) = \begin{cases} P(N(1) = 1) = e^{-1}, & \text{if } t = 1; \\ P(N(2) = 0) = e^{-2}, & \text{if } t = 2; \\ 0, & \text{if } t \neq 1, 2. \end{cases}$$

$Tf(t)$ is not a log concave function $[0, \infty)$ (the support of Tf is not an interval)

Notice that $(t + N(t),\ t \geq 0)$ has independent and stationary increments

The problem is that $t + N(t) \not\uparrow_{lr}$.

Classes of processes to be considered

Processes with stationary and independent increments ($\mathcal{IPSI}$ class):

- Gamma Process (continuous degradation in time)
- Compound Poisson models: $\sum_{i=1}^{N(t)} X_i$, with
  - $(N(t),\ t \geq 0)$: Poisson process
  - $(X_n)_{n\in\mathbb{N}^*}$: nonnegative i.i.d. random variables

  (shock models)

Processes with increments decreasing ($\mathcal{IPDI}$) or increasing ($\mathcal{IPII}$)).

To obtain processes in these classes, we consider a process $(X(t),\ t \geq 0)$ in the $\mathcal{IPSI}$ class and consider

$$X^*(t) = a_1(t) + X(a_2(t)), \quad t \geq 0.$$

- If a_i are increasing and concave: $(X^*(t),\ t \geq 0)$ is in the $\mathcal{IPDI}$ class
- If a_i are increasing and convex: $(X^*(t),\ t \geq 0)$ is in the $\mathcal{IPII}$ class