Hardy inequalities for fractional operators

Óscar Ciaurri Ramírez

Departamento de Matemáticas y Computación
Universidad de La Rioja

3rd Fractional Calculus Meeting

IUMA–Universidad de Zaragoza
Zaragoza

September 2019

The beginning: the classical Hardy inequality

The beginning: the classical Hardy inequality

- The classical Hardy inequality:

The beginning: the classical Hardy inequality

- The classical Hardy inequality: If $p > 1$,

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(t)\,dt \right|^p dx < \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p\,dx.$$

The beginning: the classical Hardy inequality

- The classical Hardy inequality: If $p > 1$,

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(t)\, dt \right|^p dx < \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p\, dx.$$

  - And the dual one:

The beginning: the classical Hardy inequality

- The classical Hardy inequality: If $p > 1$,

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(t)\, dt \right|^p dx < \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p\, dx.$$

  - And the dual one: If $p > 1$,

$$\int_0^\infty \left| \int_x^\infty f(t)\, \frac{dt}{t} \right|^p dx < p^p \int_0^\infty |f(x)|^p\, dx.$$

The beginning: the classical Hardy inequality

- The classical Hardy inequality: If $p > 1$,

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(t)\, dt \right|^p dx < \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p\, dx.$$

- And the dual one: If $p > 1$,

$$\int_0^\infty \left| \int_x^\infty f(t)\, \frac{dt}{t} \right|^p dx < p^p \int_0^\infty |f(x)|^p\, dx.$$

- **The proof:** Minkowski's integral inequality and a proper function (with a limit argument) to deduce the sharpness of the constant.

The beginning: the classical Hardy inequality

- The classical Hardy inequality: If $p > 1$,

$$\int_0^\infty \left| \frac{1}{x} \int_0^x f(t)\, dt \right|^p dx < \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p\, dx.$$

- And the dual one: If $p > 1$,

$$\int_0^\infty \left| \int_x^\infty f(t)\, \frac{dt}{t} \right|^p dx < p^p \int_0^\infty |f(x)|^p\, dx.$$

- **The proof:** Minkowski's integral inequality and a proper function (with a limit argument) to deduce the sharpness of the constant. The constant is not attained.

The beginning: the classical Hardy inequality

Note on a Theorem of Hilbert.

By

G. H. Hardy.

1. It was proved by Hilbert, in the course of his investigations in the theory of integral equations, that *the series*

$$\sum_{m,n=1}^{\infty} \frac{a_m a_n}{m+n} \qquad (a_m \geqq 0)$$

is convergent whenever $\sum a_m^2$ *is convergent.* Of this theorem, which is one of the simplest and most beautiful in the theory of double series of positive terms, at least five essentially different proofs have been published. Hilbert's own proof, which depends upon the theory of Fourier's series, is outlined by Weyl in his *Inaugural-Dissertation*[1]. Another proof was given by Wiener[2], and two more by Schur[3]; but none of these proofs is as simple and elementary as might be desired. Thus Schur's first proof depends upon the theory of quadratic and bilinear forms in an infinity of variables; and his second (which is unquestionably the most elegant of all) on a change of variables in a double integral. And Wiener's proof, while genuinely elementary, is distinctly artificial.

To these four proofs I added recently[4] a fifth which seemed to me to lack nothing in simplicity. I observed first that Hilbert's theorem is an immediate corollary of another theorem which seems of some interest in itself. This theorem is as follows:

[1]) H. Weyl, „Singuläre Integralgleichungen", Göttingen 1908, p. 83.

[2]) F. W. Wiener, „Elementarer Beweis eines Reihensatzes von Herrn Hilbert", Math. Annalen, **68** (1910), pp. 361—366.

[3]) I. Schur, „Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen", Journal für Math., **140** (1912), pp. 1—28.

[4]) G. H. Hardy, „Notes on some points in the integral calculus (51)", Messenger of Mathematics, **48** (1918), pp. 107—112.

Math.Z. 1920, p. 314

The beginning: the classical Hardy inequality

Note on a Theorem of Hilbert.

By

G. H. Hardy.

1. It was proved by Hilbert, in the course of his investigations in the theory of integral equations, that *the series*

$$\sum_{m,n=1}^{\infty} \frac{a_m a_n}{m+n} \qquad (a_m \geqq 0)$$

is convergent whenever $\sum a_m^2$ *is convergent.* Of this theorem, which is one of the simplest and most beautiful in the theory of double series of positive terms, at least five essentially different proofs have been published. Hilbert's own proof, which depends upon the theory of Fourier's series, is outlined by Weyl in his *Inaugural-Dissertation*[1]. Another proof was given by Wiener[2], and two more by Schur[3]; but none of these proofs is as simple and elementary as might be desired. Thus Schur's first proof depends upon the theory of quadratic and bilinear forms in an infinity of variables; and his second (which is unquestionably the most elegant of all) on a change of variables in a double integral. And Wiener's proof, while genuinely elementary, is distinctly artificial.

To these four proofs I added recently[4] a fifth which seemed to me to lack nothing in simplicity. I observed first that Hilbert's theorem is an immediate corollary of another theorem which seems of some interest in itself. This theorem is as follows:

---

[1]) H. Weyl, „Singuläre Integralgleichungen", Göttingen 1908, p. 83.

[2]) F. W. Wiener, „Elementarer Beweis eines Reihensatzes von Herrn Hilbert", Math. Annalen, **68** (1910), pp. 361—366.

[3]) I. Schur, „Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen", Journal für Math., **140** (1912), pp. 1—28.

[4]) G. H. Hardy, „Notes on some points in the integral calculus (51)", Messenger of Mathematics, **48** (1918), pp. 107—112.

Math.Z. 1920, p. 314

- Hardy proves the inequality for sequences to obtain an elementary proof of Hilbert's inequality.

The beginning: the classical Hardy inequality

G. H. Hardy. Note on a Theorem of Hilbert. 315

Theorem A. *If $\sum a_n^2$ is convergent then*

$$\sum \left(\frac{A_n}{n}\right)^2,$$

where $A_n = a_1 + a_2 + \ldots + a_n$, is convergent.

In order to deduce Hilbert's theorem from Theorem A, we have only to observe that the convergence of the two series in question involves that of

$$\sum \frac{a_n A_n}{n},$$

and *a fortiori* of

$$\sum_n a_n \sum_{m=1}^{n} \frac{a_m}{m+n}$$

or of

$$\sum_{m \leq n} \frac{a_m a_n}{m+n}.$$

2. Dr. Marcel Riesz, to whom recently I communicated Theorem A, at once found another proof which is equal to my own in simplicity, and which seems to both of us more natural and therefore preferable. His proof naturally suggests an interesting generalisation, viz:

Theorem B. If $\varkappa > 1$ *and $\sum a_n^\varkappa$ is convergent, then*

$$\sum \left(\frac{A_n}{n}\right)^\varkappa$$

is convergent.

I give the proof of this theorem here: Riesz's proof of Theorem A is obtained by writing $\varkappa = 2$ throughout the argument.

Let

$$\Phi_n = n^{-\varkappa} + (n+1)^{-\varkappa} + (n+2)^{-\varkappa} + \ldots.$$

Then

$$\sum_1^N \left(\frac{A_n}{n}\right)^\varkappa = \sum_1^N A_n^\varkappa (\Phi_n - \Phi_{n+1}) = \sum_1^N (A_n^\varkappa - A_{n-1}^\varkappa)\Phi_n - A_N^\varkappa \Phi_{N+1} \text{ [5)]}$$

$$\leq \sum_1^N (A_n^\varkappa - A_{n-1}^\varkappa)\Phi_n \leq \varkappa \sum_1^N a_n A_n^{\varkappa-1} \Phi_n.$$

But

(1) $$\Phi_n < n^{-\varkappa} + \int_n^\infty x^{-\varkappa}\,dx = n^{-\varkappa} + \frac{n^{-(\varkappa-1)}}{\varkappa-1} \leq \frac{\varkappa}{\varkappa-1} n^{-(\varkappa-1)};$$

5) $A_0 = 0$.

Math.Z. 1920, p. 315

• Hardy proves the inequality for sequences to obtain an elementary proof of Hilbert's inequality.

The beginning: the classical Hardy inequality

G. H. Hardy. Note on a Theorem of Hilbert. 315

Theorem A. *If $\sum a_n^2$ is convergent then*

$$\sum \left(\frac{A_n}{n}\right)^2,$$

where $A_n = a_1 + a_2 + \dots + a_n$, is convergent.

In order to deduce Hilbert's theorem from Theorem A, we have only to observe that the convergence of the two series in question involves that of

$$\sum \frac{a_n A_n}{n},$$

and *a fortiori* of

$$\sum_n a_n \sum_{m=1}^{n} \frac{a_m}{m+n}$$

or of

$$\sum_{m<n} \frac{a_m a_n}{m+n}.$$

2. Dr. Marcel Riesz, to whom recently I communicated Theorem A, at once found another proof which is equal to my own in simplicity, and which seems to both of us more natural and therefore preferable. His proof naturally suggests an interesting generalisation, viz:

Theorem B. If $\varkappa > 1$ *and $\sum a_n^\varkappa$ is convergent, then*

$$\sum \left(\frac{A_n}{n}\right)^\varkappa$$

is convergent.

I give the proof of this theorem here: Riesz's proof of Theorem A is obtained by writing $\varkappa = 2$ throughout the argument.

Let

$$\Phi_n = n^{-\varkappa} + (n+1)^{-\varkappa} + (n+2)^{-\varkappa} + \dots .$$

Then

$$\sum_1^N \left(\frac{A_n}{n}\right)^\varkappa = \sum_1^N A_n^\varkappa (\Phi_n - \Phi_{n+1}) = \sum_1^N (A_n^\varkappa - A_{n-1}^\varkappa)\Phi_n - A_N^\varkappa \Phi_{N+1}\ {}^{5)}$$

$$\leqq \sum_1^N (A_n^\varkappa - A_{n-1}^\varkappa)\Phi_n \leqq \varkappa \sum_1^N a_n A_n^{\varkappa-1}\Phi_n .$$

But

(1) $$\Phi_n < n^{-\varkappa} + \int_n^\infty x^{-\varkappa}\,dx = n^{-\varkappa} + \frac{n^{-(\varkappa-1)}}{\varkappa-1} \leqq \frac{\varkappa}{\varkappa-1} n^{-(\varkappa-1)};$$

[5] $A_0 = 0$.

Math.Z. 1920, p. 315

- Hardy proves the inequality for sequences to obtain an elementary proof of Hilbert's inequality.
- Hilbert's result follows from the case $\kappa = 2$ but he obtains the inequality for any $\kappa > 1$.

The beginning: the classical Hardy inequality

The corresponding inequality for integrals is

$$\int_a^\infty \left(\frac{F}{x}\right)^\kappa dx \leqq \left(\frac{\kappa}{\kappa-1}\right)^\kappa \int_a^\infty f^\kappa dx. \tag{4}$$

Here

$$F = \int_a^x f\,dt,$$

and a and f are positive. It will be observed that in this inequality we have a smaller constant than in (1), and it is easy to show, by taking $f = x^{-\frac{1}{\kappa}-\varepsilon}$, where ε is small, that the constant is in fact the correct one. I can show that, in (2), the constant cannot be replaced by any smaller constant than that which occurs in (4); but whether this constant will suffice I cannot say.

Math.Z. 1920, p. 317

• Hardy proves the inequality for sequences to obtain an elementary proof of Hilbert's inequality.

• Hilbert's result follows from the case $\kappa = 2$ but he obtains the inequality for any $\kappa > 1$.

The beginning: the classical Hardy inequality

The corresponding inequality for integrals is

$$(4) \qquad \int_a^\infty \left(\frac{F}{x}\right)^\kappa dx \leq \left(\frac{\kappa}{\kappa-1}\right)^\kappa \int_a^\infty f^\kappa \, dx.$$

Here

$$F = \int_a^x f \, dt,$$

and a and f are positive. It will be observed that in this inequality we have a smaller constant than in (1), and it is easy to show, by taking $f = x^{-\frac{1}{\kappa}-\varepsilon}$, where ε is small, that the constant is in fact the correct one. I can show that, in (2), the constant cannot be replaced by any smaller constant than that which occurs in (4); but whether this constant will suffice I cannot say.

Math.Z. 1920, p. 317

- Hardy proves the inequality for sequences to obtain an elementary proof of Hilbert's inequality.
- Hilbert's result follows from the case $\kappa = 2$ but he obtains the inequality for any $\kappa > 1$.
- Hardy gives the extension for integrals.

The beginning: the classical Hardy inequality

The corresponding inequality for integrals is

$$\int_a^\infty \left(\frac{F}{x}\right)^\kappa dx \leq \left(\frac{\kappa}{\kappa-1}\right)^\kappa \int_a^\infty f^\kappa dx. \qquad (4)$$

Here

$$F = \int_a^x f\,dt,$$

and a and f are positive. It will be observed that in this inequality we have a smaller constant than in (1), and it is easy to show, by taking $f = x^{-\frac{1}{\kappa}-\varepsilon}$, where ε is small, that the constant is in fact the correct one. I can show that, in (2), the constant cannot be replaced by any smaller constant than that which occurs in (4); but whether this constant will suffice I cannot say.

Math.Z. 1920, p. 317

- Hardy proves the inequality for sequences to obtain an elementary proof of Hilbert's inequality.
- Hilbert's result follows from the case $\kappa = 2$ but he obtains the inequality for any $\kappa > 1$.
- Hardy gives the extension for integrals.
- The given constant is sharp for the integral inequality but not in the inequality for sequences.

The beginning: the classical Hardy inequality

The corresponding inequality for integrals is

$$\int_a^\infty \left(\frac{F}{x}\right)^\kappa dx \leqq \left(\frac{\kappa}{\kappa-1}\right)^\kappa \int_a^\infty f^\kappa dx. \tag{4}$$

Here

$$F = \int_a^x f\,dt,$$

and a and f are positive. It will be observed that in this inequality we have a smaller constant than in (1), and it is easy to show, by taking $f = x^{-\frac{1}{\kappa}-\varepsilon}$, where ε is small, that the constant is in fact the correct one. I can show that, in (2), the constant cannot be replaced by any smaller constant than that which occurs in (4); but whether this constant will suffice I cannot say.

Math.Z. 1920, p. 317

- Hardy proves the inequality for sequences to obtain an elementary proof of Hilbert's inequality.
- Hilbert's result follows from the case $\kappa = 2$ but he obtains the inequality for any $\kappa > 1$.
- Hardy gives the extension for integrals.
- The given constant is sharp for the integral inequality but not in the inequality for sequences.
- A detailed proof, by using integration by parts, can be found in *Inequalities*.

The beginning: the classical Hardy inequality

The corresponding inequality for integrals is

$$\int_a^\infty \left(\frac{F}{x}\right)^\varkappa dx \leqq \left(\frac{\varkappa}{\varkappa-1}\right)^\varkappa \int_a^\infty f^\varkappa dx. \qquad (4)$$

Here

$$F = \int_a^x f\,dt,$$

and a and f are positive. It will be observed that in this inequality we have a smaller constant than in (1), and it is easy to show, by taking $f = x^{-\frac{1}{\varkappa}-\varepsilon}$, where ε is small, that the constant is in fact the correct one. I can show that, in (2), the constant cannot be replaced by any smaller constant than that which occurs in (4); but whether this constant will suffice I cannot say.

Math.Z. 1920, p. 317

- Hardy proves the inequality for sequences to obtain an elementary proof of Hilbert's inequality.
- Hilbert's result follows from the case $\kappa = 2$ but he obtains the inequality for any $\kappa > 1$.
- Hardy gives the extension for integrals.
- The given constant is sharp for the integral inequality but not in the inequality for sequences.
- A detailed proof, by using integration by parts, can be found in *Inequalities*.
- A. Kufner, L. Maligranda and L. E. Persson, The Prehistory of the Hardy Inequality, *Amer. Math. Monthly*, 2006.

The Hardy inequality for the fractional Laplacian

The Hardy inequality for the fractional Laplacian

- For a continuous function f, taking $p = 2$ and

$$u(x) = \int_0^x f(t)\,dt,$$

The Hardy inequality for the fractional Laplacian

- For a continuous function f, taking $p = 2$ and

$$u(x) = \int_0^x f(t)\, dt,$$

the classical Hardy inequality becomes

$$\int_0^\infty |u'(x)|^2\, dx > \frac{1}{4} \int_0^\infty \frac{|u(x)|^2}{x^2}\, dx.$$

The Hardy inequality for the fractional Laplacian

- For a continuous function f, taking $p = 2$ and

$$u(x) = \int_0^x f(t)\,dt,$$

the classical Hardy inequality becomes

$$\int_0^\infty |u'(x)|^2\,dx > \frac{1}{4}\int_0^\infty \frac{|u(x)|^2}{x^2}\,dx.$$

- For nice functions and $d \geq 3$

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2\,dx > \frac{(d-2)^2}{4}\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\,dx,$$

The Hardy inequality for the fractional Laplacian

- For a continuous function f, taking $p = 2$ and

$$u(x) = \int_0^x f(t)\,dt,$$

the classical Hardy inequality becomes

$$\int_0^\infty |u'(x)|^2\,dx > \frac{1}{4}\int_0^\infty \frac{|u(x)|^2}{x^2}\,dx.$$

- For nice functions and $d \geq 3$

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2\,dx > \frac{(d-2)^2}{4}\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\,dx,$$

or, equivalently,

$$\int_{\mathbb{R}^d} \overline{u}(x)(-\Delta)u(x)\,dx > \frac{(d-2)^2}{4}\int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2}\,dx.$$

The Hardy inequality for the fractional Laplacian

The Hardy inequality for the fractional Laplacian

- A proof with integration by parts

$$
\begin{aligned}
0 &\leq \int_{\mathbb{R}^d} \left| \nabla u(x) - \alpha \frac{x}{|x|} u(x) \right|^2 \, dx \\
&= \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx - \alpha \int_{\mathbb{R}^d} \left\langle \nabla u(x), x \frac{u(x)}{|x|} \right\rangle + \alpha^2 \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} \, dx \\
&= \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx + ((d-2)\alpha + \alpha^2) \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} \, dx.
\end{aligned}
$$

The Hardy inequality for the fractional Laplacian

- A proof with integration by parts

$$
\begin{aligned}
0 &\leq \int_{\mathbb{R}^d} \left| \nabla u(x) - \alpha \frac{x}{|x|} u(x) \right|^2 \, dx \\
&= \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx - \alpha \int_{\mathbb{R}^d} \left\langle \nabla u(x), x \frac{u(x)}{|x|} \right\rangle + \alpha^2 \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} \, dx \\
&= \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx + ((d-2)\alpha + \alpha^2) \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} \, dx.
\end{aligned}
$$

- The minimum is attained for $\alpha = -(d-2)/2$ and in that case $(d-2)\alpha + \alpha^2 = -(d-2)^2/4$.

The Hardy inequality for the fractional Laplacian

- A proof with integration by parts

$$
\begin{aligned}
0 &\le \int_{\mathbb{R}^d} \left| \nabla u(x) - \alpha \frac{x}{|x|} u(x) \right|^2 \, dx \\
&= \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx - \alpha \int_{\mathbb{R}^d} \left\langle \nabla u(x), x \frac{u(x)}{|x|} \right\rangle + \alpha^2 \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} \, dx \\
&= \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx + ((d-2)\alpha + \alpha^2) \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} \, dx.
\end{aligned}
$$

- The minimum is attained for $\alpha = -(d-2)/2$ and in that case $(d-2)\alpha + \alpha^2 = -(d-2)^2/4$.
- The constant is optimal but not attained.

The Hardy inequality for the fractional Laplacian

The Hardy inequality for the fractional Laplacian

- For $\sigma > 0$ we consider the operator $(-\Delta)^\sigma$ given by $\widehat{(-\Delta)^\sigma u} = |\cdot|^{2\sigma}\hat{u}$.

The Hardy inequality for the fractional Laplacian

- For $\sigma > 0$ we consider the operator $(-\Delta)^\sigma$ given by $\widehat{(-\Delta)^\sigma u} = |\cdot|^{2\sigma}\hat{u}$.
- For $0 < \sigma < \min\{1, d/2\}$, the inequality

$$h[u] := \int_{\mathbb{R}^d} \overline{u}(x)(-\Delta)^\sigma u(x)\, dx - C_\sigma \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^{2\sigma}}\, dx \geq 0$$

holds, with

$$C_\sigma = \left(2^\sigma \frac{\Gamma((d+2\sigma)/4)}{\Gamma((d-2\sigma)/4)}\right)^2.$$

The Hardy inequality for the fractional Laplacian

- For $\sigma > 0$ we consider the operator $(-\Delta)^\sigma$ given by $\widehat{(-\Delta)^\sigma u} = |\cdot|^{2\sigma}\hat{u}$.
- For $0 < \sigma < \min\{1, d/2\}$, the inequality

$$h[u] := \int_{\mathbb{R}^d} \overline{u}(x)(-\Delta)^\sigma u(x)\,dx - C_\sigma \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^{2\sigma}}\,dx \geq 0$$

holds, with

$$C_\sigma = \left(2^\sigma \frac{\Gamma((d+2\sigma)/4)}{\Gamma((d-2\sigma)/4)}\right)^2.$$

- **First ingredient**: for $0 < \alpha < d$,

$$b_\alpha \widehat{|\cdot|^{-\alpha}}(\xi) = b_{d-\alpha}|\xi|^{-d+\alpha}, \qquad b_\alpha := 2^{\alpha/2}\Gamma(\alpha/2).$$

The Hardy inequality for the fractional Laplacian

- For $\sigma > 0$ we consider the operator $(-\Delta)^\sigma$ given by $\widehat{(-\Delta)^\sigma u} = |\cdot|^{2\sigma}\hat{u}$.
- For $0 < \sigma < \min\{1, d/2\}$, the inequality

$$h[u] := \int_{\mathbb{R}^d} \overline{u}(x)(-\Delta)^\sigma u(x)\,dx - C_\sigma \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^{2\sigma}}\,dx \geq 0$$

holds, with

$$C_\sigma = \left(2^\sigma \frac{\Gamma((d+2\sigma)/4)}{\Gamma((d-2\sigma)/4)}\right)^2.$$

- **First ingredient**: for $0 < \alpha < d$,

$$b_\alpha \widehat{|\cdot|^{-\alpha}}(\xi) = b_{d-\alpha}|\xi|^{-d+\alpha}, \qquad b_\alpha := 2^{\alpha/2}\Gamma(\alpha/2).$$

- Then, for $0 < \alpha < d - 2\sigma$,

$$(-\Delta)^\sigma |\cdot|^{-\alpha} = 2^{2\sigma} \frac{\Gamma((d-\alpha)/2)}{\Gamma((d-\alpha)/2-\sigma)} \frac{\Gamma(\alpha/2+\sigma)}{\Gamma(\alpha/2)} |\cdot|^{-\alpha-2\sigma}.$$

The Hardy inequality for the fractional Laplacian

The Hardy inequality for the fractional Laplacian

- **Second ingredient**: for $0 < \sigma < 1$

$$\int_{\mathbb{R}^d} \overline{f}(x)(-\Delta)^\sigma f(x)\,dx = c_{d,\sigma} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2\sigma}}\,dy,$$

con $c_{d,\sigma} = \frac{2^{2\sigma-1}}{\pi^{d/2}} \frac{\Gamma(\sigma+d/2)}{|\Gamma(-\sigma)|}$.

The Hardy inequality for the fractional Laplacian

- **Second ingredient**: for $0 < \sigma < 1$

$$\int_{\mathbb{R}^d} \overline{f}(x)(-\Delta)^\sigma f(x)\,dx = c_{d,\sigma} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2\sigma}}\,dy,$$

con $c_{d,\sigma} = \frac{2^{2\sigma-1}}{\pi^{d/2}}\frac{\Gamma(\sigma+d/2)}{|\Gamma(-\sigma)|}$.

- Then

$$\int_{\mathbb{R}^d} \overline{g}(x)(-\Delta)^\sigma f(x)\,dx = c_{d,\sigma} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(f(x)-f(y))(\overline{g}(x)-\overline{g}(y))}{|x-y|^{d+2\sigma}}\,dx\,dy.$$

The Hardy inequality for the fractional Laplacian

- **Second ingredient**: for $0 < \sigma < 1$

$$\int_{\mathbb{R}^d} \overline{f}(x)(-\Delta)^\sigma f(x)\,dx = c_{d,\sigma}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2\sigma}}\,dy,$$

con $c_{d,\sigma} = \frac{2^{2\sigma-1}}{\pi^{d/2}}\frac{\Gamma(\sigma+d/2)}{|\Gamma(-\sigma)|}$.

- Then

$$\int_{\mathbb{R}^d} \overline{g}(x)(-\Delta)^\sigma f(x)\,dx = c_{d,\sigma}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(f(x)-f(y))(\overline{g}(x)-\overline{g}(y))}{|x-y|^{d+2\sigma}}\,dx\,dy.$$

- Taking $g(x) = |x|^{-(d-2\sigma)/2}$ and $f(x) = |u(x)|^2/g(x)$ we obtain

$$h[u] = a_\sigma \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|v(x)-v(y)|^2}{|x-y|^{d+2\sigma}}\frac{dx}{|x|^{(d-2\sigma)/2}}\frac{dy}{|y|^{(d-2\sigma)/2}},$$

with $v(x) = |x|^{(d-2\sigma)/2}u(x)$ and the Hardy inequality is obtained.

The Hardy inequality for the fractional Laplacian

- **Second ingredient**: for $0 < \sigma < 1$

$$\int_{\mathbb{R}^d} \overline{f}(x)(-\Delta)^\sigma f(x)\,dx = c_{d,\sigma} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2\sigma}}\,dy,$$

con $c_{d,\sigma} = \frac{2^{2\sigma-1}}{\pi^{d/2}}\frac{\Gamma(\sigma+d/2)}{|\Gamma(-\sigma)|}$.

- Then

$$\int_{\mathbb{R}^d} \overline{g}(x)(-\Delta)^\sigma f(x)\,dx = c_{d,\sigma} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(f(x)-f(y))(\overline{g}(x)-\overline{g}(y))}{|x-y|^{d+2\sigma}}\,dx\,dy.$$

- Taking $g(x) = |x|^{-(d-2\sigma)/2}$ and $f(x) = |u(x)|^2/g(x)$ we obtain

$$h[u] = a_\sigma \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|v(x)-v(y)|^2}{|x-y|^{d+2\sigma}}\frac{dx}{|x|^{(d-2\sigma)/2}}\frac{dy}{|y|^{(d-2\sigma)/2}},$$

with $v(x) = |x|^{(d-2\sigma)/2}u(x)$ and the Hardy inequality is obtained.

- *Ground state representation*, Frank, Lieb and Seiringer (2007) (Loss in a particular case).

The Hardy inequality for the fractional Laplacian

The Hardy inequality for the fractional Laplacian

• The Hardy inequality for the fractional Laplacian was previously proved with different techniques by Herbst (Comm. Math. Phys. 1977), Beckner (PAMS 1991), and Yafaev (JFA 1999).

The Hardy inequality for the fractional Laplacian

• The Hardy inequality for the fractional Laplacian was previously proved with different techniques by Herbst (Comm. Math. Phys. 1977), Beckner (PAMS 1991), and Yafaev (JFA 1999).

• By using Yafaev's technique, Gorbachev, Ivanov, and Tikhonov have obtained the Hardy inequality for the Dunkl Laplacian (JAT, 2016) and its a-deformed version (IMRN, 2016).

The Hardy inequality for the fractional Laplacian

• The Hardy inequality for the fractional Laplacian was previously proved with different techniques by Herbst (Comm. Math. Phys. 1977), Beckner (PAMS 1991), and Yafaev (JFA 1999).
• By using Yafaev's technique, Gorbachev, Ivanov, and Tikhonov have obtained the Hardy inequality for the Dunkl Laplacian (JAT, 2016) and its a-deformed version (IMRN, 2016).
• With adequate versions of **First ingredient** and **Second ingredient**, we can deduce other Hardy inequalities.

The Hardy inequality for the fractional Laplacian

• The Hardy inequality for the fractional Laplacian was previously proved with different techniques by Herbst (Comm. Math. Phys. 1977), Beckner (PAMS 1991), and Yafaev (JFA 1999).
• By using Yafaev's technique, Gorbachev, Ivanov, and Tikhonov have obtained the Hardy inequality for the Dunkl Laplacian (JAT, 2016) and its a-deformed version (IMRN, 2016).
• With adequate versions of **First ingredient** and **Second ingredient**, we can deduce other Hardy inequalities.
• For the sublaplacian of the Heisenberg group: Roncal and Thangavelu (AIM, 2016).

The Hardy inequality for the fractional Laplacian

• The Hardy inequality for the fractional Laplacian was previously proved with different techniques by Herbst (Comm. Math. Phys. 1977), Beckner (PAMS 1991), and Yafaev (JFA 1999).
• By using Yafaev's technique, Gorbachev, Ivanov, and Tikhonov have obtained the Hardy inequality for the Dunkl Laplacian (JAT, 2016) and its a-deformed version (IMRN, 2016).
• With adequate versions of **First ingredient** and **Second ingredient**, we can deduce other Hardy inequalities.
• For the sublaplacian of the Heisenberg group: Roncal and Thangavelu (AIM, 2016).
• For the harmonic oscillator in the Dunkl setting: C., Roncal, and Thangavelu (PEMS, 2018).

The Hardy inequality for the fractional Laplacian

• The Hardy inequality for the fractional Laplacian was previously proved with different techniques by Herbst (Comm. Math. Phys. 1977), Beckner (PAMS 1991), and Yafaev (JFA 1999).
• By using Yafaev's technique, Gorbachev, Ivanov, and Tikhonov have obtained the Hardy inequality for the Dunkl Laplacian (JAT, 2016) and its *a*-deformed version (IMRN, 2016).
• With adequate versions of **First ingredient** and **Second ingredient**, we can deduce other Hardy inequalities.
• For the sublaplacian of the Heisenberg group: Roncal and Thangavelu (AIM, 2016).
• For the harmonic oscillator in the Dunkl setting: C., Roncal, and Thangavelu (PEMS, 2018).
• For the discrete Laplacian: C. and Roncal (Journal of Analysis, 2018).

The Hardy inequality for ultraspherical expansions

The Hardy inequality for ultraspherical expansions

• For $\lambda > -1/2$, let $\{c_n^\lambda\}_{n\geq 0}$ be the sequence of ultraspherical polynomials normalized in $L^2((-1,1), d\mu_\lambda)$, with

$$d\mu_\lambda(x) = (1-x^2)^{\lambda-1/2}\,dx.$$

The Hardy inequality for ultraspherical expansions

• For $\lambda > -1/2$, let $\{c_n^\lambda\}_{n\geq 0}$ be the sequence of ultraspherical polynomials normalized in $L^2((-1,1), d\mu_\lambda)$, with

$$d\mu_\lambda(x) = (1-x^2)^{\lambda-1/2}\,dx.$$

• For each function f in $(-1,1)$, we define its ultraspherical expansion

$$f \longmapsto \sum_{n=0}^{\infty} a_n(f) c_n^\lambda, \qquad a_n(f) = \int_{-1}^{1} f(x) c_n^\lambda(x)\,d\mu_\lambda(x).$$

The Hardy inequality for ultraspherical expansions

• For $\lambda > -1/2$, let $\{c_n^\lambda\}_{n\geq 0}$ be the sequence of ultraspherical polynomials normalized in $L^2((-1,1), d\mu_\lambda)$, with

$$d\mu_\lambda(x) = (1-x^2)^{\lambda-1/2}\, dx.$$

• For each function f in $(-1,1)$, we define its ultraspherical expansion

$$f \longmapsto \sum_{n=0}^{\infty} a_n(f) c_n^\lambda, \qquad a_n(f) = \int_{-1}^{1} f(x) c_n^\lambda(x)\, d\mu_\lambda(x).$$

• For each $\sigma > 0$, we define the operator

$$\mathbf{A}_\sigma^\lambda f(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} a_n(f) c_n^\lambda(x).$$

The Hardy inequality for ultraspherical expansions

The Hardy inequality for ultraspherical expansions

- The natural Sobolev space to analyse Hardy type inequalities is

$$H_\lambda^\sigma = \left\{ f : \|f\|_{H_\lambda^\sigma} := \left(\sum_{n=0}^{\infty} (n+\lambda)^\sigma (a_n(f))^2 \right)^{1/2} < \infty \right\}.$$

The Hardy inequality for ultraspherical expansions

- The natural Sobolev space to analyse Hardy type inequalities is

$$H_\lambda^\sigma = \left\{ f : \|f\|_{H_\lambda^\sigma} := \left(\sum_{n=0}^{\infty} (n+\lambda)^\sigma (a_n(f))^2 \right)^{1/2} < \infty \right\}.$$

Theorem

Let $\lambda > 0$ *and* $0 < \sigma < 1$. *Then for* $u \in H_\lambda^\sigma$

$$Q_{\sigma,\lambda} \int_{-1}^{1} \frac{u^2(x)}{(1-x^2)^{\sigma/2}}\, d\mu_\lambda(x) \le \int_{-1}^{1} u(x) \mathbf{A}_\sigma^\lambda u(x)\, d\mu_\lambda(x),$$

where

$$Q_{\sigma,\lambda} = 2^\sigma \left(\frac{\Gamma(\frac{\lambda}{2} + \frac{1+\sigma}{4})}{\Gamma(\frac{\lambda}{2} + \frac{1-\sigma}{4})} \right)^2 .$$

The Hardy inequality for ultraspherical expansions

The Hardy inequality for ultraspherical expansions

- Hardy inequality can be rewritten in terms of the Fourier coefficients

$$Q_{\sigma,\lambda}\int_{-1}^{1}\frac{u^2(x)}{(1-x^2)^{\sigma/2}}\,d\mu_\lambda(x)\leq\sum_{n=0}^{\infty}\frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})}(a_n(u))^2,$$

which is the Pitt inequality for the ultraspherical expansions

The Hardy inequality for ultraspherical expansions

- Hardy inequality can be rewritten in terms of the Fourier coefficients

$$Q_{\sigma,\lambda}\int_{-1}^{1}\frac{u^2(x)}{(1-x^2)^{\sigma/2}}\,d\mu_\lambda(x)\le\sum_{n=0}^{\infty}\frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})}(a_n(u))^2,$$

which is the Pitt inequality for the ultraspherical expansions

- An uncertainty principle:

Corollary

Let $\lambda>0$ and $0<\sigma<1$. Then for $u\in H_\lambda^\sigma$

$$Q_{\sigma,\lambda}\left(\int_{-1}^{1}u^2(x)\,d\mu_\lambda(x)\right)^2\le\int_{-1}^{1}u^2(x)(1-x^2)^{\sigma/2}\,d\mu_\lambda(x)\int_{-1}^{1}u(x)\mathbf{A}_\sigma^\lambda u(x)\,d\mu_\lambda(x),$$

where $Q_{\sigma,\lambda}$ is the constant in the Hardy inequality.

The Hardy inequality for ultraspherical expansions

The Hardy inequality for ultraspherical expansions

• The Pitt inequality allows us to prove a logarithmic uncertainty principle for the ultraspherical expansions: if $\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}$, we have

$$\left(\log 2 + \psi\left(\frac{\lambda}{2} + \frac{1}{4}\right)\right) \int_{-1}^{1} u^2(x)\, d\mu_\lambda(x)$$
$$\leq \sum_{n=0}^{\infty} \psi\left(n + \lambda + \frac{1}{2}\right) (a_n(u))^2 + \int_{-1}^{1} \log(\sqrt{1-x^2}) u^2(x)\, d\mu_\lambda(x).$$

The Hardy inequality for ultraspherical expansions

• The Pitt inequality allows us to prove a logarithmic uncertainty principle for the ultraspherical expansions: if $\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}$, we have

$$\left(\log 2 + \psi\left(\frac{\lambda}{2} + \frac{1}{4}\right)\right) \int_{-1}^{1} u^2(x)\, d\mu_\lambda(x) \\ \leq \sum_{n=0}^{\infty} \psi\left(n + \lambda + \frac{1}{2}\right) (a_n(u))^2 + \int_{-1}^{1} \log(\sqrt{1-x^2}) u^2(x)\, d\mu_\lambda(x).$$

• The proof follows Beckner (1995). For a derivable function such that $\phi(0) = 0$ and $\phi(\sigma) > 0$ for $\sigma \in (0, \varepsilon)$, with $\varepsilon > 0$, it is verified that $\phi'(0_+) > 0$.

The Hardy inequality for ultraspherical expansions

• The Pitt inequality allows us to prove a logarithmic uncertainty principle for the ultraspherical expansions: if $\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}$, we have

$$\left(\log 2 + \psi\left(\frac{\lambda}{2} + \frac{1}{4}\right)\right) \int_{-1}^{1} u^2(x)\, d\mu_\lambda(x) \leq \sum_{n=0}^{\infty} \psi\left(n + \lambda + \frac{1}{2}\right)(a_n(u))^2 + \int_{-1}^{1} \log(\sqrt{1-x^2}) u^2(x)\, d\mu_\lambda(x).$$

• The proof follows Beckner (1995). For a derivable function such that $\phi(0) = 0$ and $\phi(\sigma) > 0$ for $\sigma \in (0, \varepsilon)$, with $\varepsilon > 0$, it is verified that $\phi'(0_+) > 0$.

• Taking the function

$$\phi(\sigma) = \sum_{n=0}^{\infty} \frac{\Gamma(n + \lambda + \frac{1+\sigma}{2})}{\Gamma(n + \lambda + \frac{1-\sigma}{2})}(a_n(u))^2 - Q_{\sigma,\lambda} \int_{-1}^{1} \frac{u^2(x)}{(1-x^2)^{\sigma/2}}\, d\mu_\lambda(x),$$

we have $\phi(0) = 0$ (the Parseval identity) and $\phi(\sigma) > 0$ for $\sigma \in (0, 1)$ (the Pitt inequality), then $\phi'(0_+) > 0$ and this is the logarithmic uncertainty principle.

The first ingredient

The first ingredient

Lemma

Let $\lambda > 0$ *and* $2\lambda + 1 > \sigma > 0$. *Then*

$$\mathbf{A}_{\sigma}^{\lambda}\left(\frac{1}{(1-x^2)^{\lambda/2+(1-\sigma)/4}}\right) = \frac{Q_{\sigma,\lambda}}{(1-x^2)^{\lambda/2+(1+\sigma)/4}},$$

where $Q_{\sigma,\lambda}$ *is the constant given in the Hardy inequality.*

The first ingredient

Lemma

Let $\lambda > 0$ *and* $2\lambda + 1 > \sigma > 0$. *Then*

$$\mathbf{A}_\sigma^\lambda\left(\frac{1}{(1-x^2)^{\lambda/2+(1-\sigma)/4}}\right) = \frac{Q_{\sigma,\lambda}}{(1-x^2)^{\lambda/2+(1+\sigma)/4}},$$

where $Q_{\sigma,\lambda}$ *is the constant given in the Hardy inequality.*

• The proof is obtained evaluating some integrals containing ultraspherical polynomials and manipulating some ${}_3F_2$ involve in them.

The second ingredient

The second ingredient

Lemma

Let $\lambda > 0$, $0 < \sigma < 1$, *and* $f \in H_\lambda^\sigma$. *Then*

$$\mathbf{A}_\sigma^\lambda f(x) = \int_{-1}^{1} (f(x) - f(y))\, K_\sigma^\lambda(x,y)\, d\mu_\lambda(y) + E_{\sigma,\lambda} f(x),$$

where the kernel is given by

$$K_\sigma^\lambda(x,y) = D_{\sigma,\lambda} \int_{-1}^{1} \frac{d\mu_\lambda(t)}{(1 - xy - \sqrt{1-x^2}\sqrt{1-y^2}\,t)^{\lambda+(1+\sigma)/2}},$$

with

$$D_{\sigma,\lambda} = \frac{c_\lambda^2}{2^{\lambda+(1+\sigma)/2}} \frac{\Gamma(\frac{1-\sigma}{2})\Gamma(\lambda + \frac{1+\sigma}{2})}{|\Gamma(-\sigma)|\Gamma(1+\lambda)}, \qquad E_{\sigma,\lambda} = \frac{\Gamma(\lambda + \frac{1+\sigma}{2})}{\Gamma(\lambda + \frac{1-\sigma}{2})}.$$

The Hardy inequality on the sphere

The Hardy inequality on the sphere

- For a function f in the sphere, we have

$$f(x) = \sum_{k=0}^{\infty} \operatorname{proj}_k f(x),$$

where $\operatorname{proj}_k f$ is the projection of the function f onto $\mathcal{H}_k^d$, the space of spherical harmonics of degree k, and it is given by

$$\operatorname{proj}_k f(x) = \int_{\mathbb{S}^{d-1}} f(y) Z_k(x,y) \frac{d\omega(y)}{s_d}, \qquad Z_k(x,y) = \sum_{j=1}^{\dim \mathcal{H}_k^d} Y_j(x) Y_j(y),$$

for $\{Y_j \,:\, 1 \le j \le \dim \mathcal{H}_k^d\}$ an orthonormal basis of $\mathcal{H}_k^d$.

The Hardy inequality on the sphere

- For a function f in the sphere, we have

$$f(x) = \sum_{k=0}^{\infty} \operatorname{proj}_k f(x),$$

where $\operatorname{proj}_k f$ is the projection of the function f onto $\mathcal{H}_k^d$, the space of spherical harmonics of degree k, and it is given by

$$\operatorname{proj}_k f(x) = \int_{\mathbb{S}^{d-1}} f(y) Z_k(x,y) \frac{d\omega(y)}{s_d}, \qquad Z_k(x,y) = \sum_{j=1}^{\dim \mathcal{H}_k^d} Y_j(x) Y_j(y),$$

for $\{Y_j : 1 \le j \le \dim \mathcal{H}_k^d\}$ an orthonormal basis of $\mathcal{H}_k^d$.

- Taking the operator

$$L_d = \Delta_{\mathbb{S}^{d-1}} - ((d-2)/2)^2$$

where $-\Delta_{\mathbb{S}^{d-1}}$ is the Laplace-Beltrami operator on the sphere, we have

$$-L_d Y_j(x) = (k + (d-2)/2)^2 Y_j(x),$$

for each spherical harmonic of degree k.

The Hardy inequality on the sphere

The Hardy inequality on the sphere

- The analogous of the operator $\mathbf{A}_\sigma^\lambda$ on $\mathbb{S}^d$ is defined by

$$A_\sigma f = \frac{\Gamma\left(\sqrt{-L_d}+\frac{1+\sigma}{2}\right)}{\Gamma\left(\sqrt{-L_d}+\frac{1-\sigma}{2}\right)} f = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{d-2}{2}+\frac{1+\sigma}{2}\right)}{\Gamma\left(n+\frac{d-2}{2}+\frac{1-\sigma}{2}\right)} \operatorname{proj}_n f,$$

The Hardy inequality on the sphere

- The analogous of the operator $\mathbf{A}_\sigma^\lambda$ on $\mathbb{S}^d$ is defined by

$$A_\sigma f = \frac{\Gamma\left(\sqrt{-L_d}+\frac{1+\sigma}{2}\right)}{\Gamma\left(\sqrt{-L_d}+\frac{1-\sigma}{2}\right)} f = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{d-2}{2}+\frac{1+\sigma}{2}\right)}{\Gamma\left(n+\frac{d-2}{2}+\frac{1-\sigma}{2}\right)} \operatorname{proj}_n f,$$

- We consider the Sobolev space

$$\mathbf{H}^\sigma = \left\{ f : \|f\|_{\mathbf{H}^\sigma} := \left(\sum_{n=0}^{\infty} \left(n+\frac{d-1}{2}\right)^\sigma \|\operatorname{proj}_n f\|^2_{L^2(\mathbb{S}^{d-1})} \right)^{1/2} < \infty \right\}.$$

The Hardy inequality on the sphere

- The analogous of the operator $\mathbf{A}_\sigma^\lambda$ on $\mathbb{S}^d$ is defined by

$$A_\sigma f = \frac{\Gamma\left(\sqrt{-L_d} + \frac{1+\sigma}{2}\right)}{\Gamma\left(\sqrt{-L_d} + \frac{1-\sigma}{2}\right)} f = \sum_{n=0}^{\infty} \frac{\Gamma\left(n + \frac{d-2}{2} + \frac{1+\sigma}{2}\right)}{\Gamma\left(n + \frac{d-2}{2} + \frac{1-\sigma}{2}\right)} \operatorname{proj}_n f,$$

- We consider the Sobolev space

$$\mathbf{H}^\sigma = \left\{ f : \|f\|_{\mathbf{H}^\sigma} := \left(\sum_{n=0}^{\infty} \left(n + \frac{d-1}{2}\right)^\sigma \|\operatorname{proj}_n f\|^2_{L^2(\mathbb{S}^{d-1})} \right)^{1/2} < \infty \right\}.$$

Theorem

Let $d \geq 3$, $0 < \sigma < 1$, and e_d be the north pole of the sphere $\mathbb{S}^{d-1}$. Then for $f \in \mathbf{H}^\sigma$

$$2^\sigma Q_{\sigma,(d-2)/2} \int_{\mathbb{S}^{d-1}} \frac{|f(x)|^2}{(|x - e_d||x + e_d|)^\sigma}\, dx \leq \int_{\mathbb{S}^{d-1}} \overline{f}(x) A_\sigma f(x)\, dx.$$

A discrete Hardy inequality in the Jacobi setting

A discrete Hardy inequality in the Jacobi setting

- For $\alpha, \beta > -1$, we consider the sequences

$$a_n^{(\alpha,\beta)} = \frac{2}{2n+2+\alpha+\beta}\sqrt{\frac{(n+1)(n+1+\alpha)(n+1+\beta)(n+1+\alpha+\beta)}{(2n+1+\alpha+\beta)(2n+3+\alpha+\beta)}}$$

and

$$b_n^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)},$$

A discrete Hardy inequality in the Jacobi setting

- For $\alpha, \beta > -1$, we consider the sequences

$$a_n^{(\alpha,\beta)} = \frac{2}{2n+2+\alpha+\beta}\sqrt{\frac{(n+1)(n+1+\alpha)(n+1+\beta)(n+1+\alpha+\beta)}{(2n+1+\alpha+\beta)(2n+3+\alpha+\beta)}}$$

and

$$b_n^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)},$$

and the operator

$$J^{(\alpha,\beta)}f(n) = a_n^{(\alpha,\beta)}f(n+1) + b_n^{(\alpha,\beta)}f(n) + a_{n-1}^{(\alpha,\beta)}f(n-1).$$

A discrete Hardy inequality in the Jacobi setting

- For $\alpha, \beta > -1$, we consider the sequences

$$a_n^{(\alpha,\beta)} = \frac{2}{2n+2+\alpha+\beta}\sqrt{\frac{(n+1)(n+1+\alpha)(n+1+\beta)(n+1+\alpha+\beta)}{(2n+1+\alpha+\beta)(2n+3+\alpha+\beta)}}$$

and

$$b_n^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)},$$

and the operator

$$J^{(\alpha,\beta)}f(n) = a_n^{(\alpha,\beta)}f(n+1) + b_n^{(\alpha,\beta)}f(n) + a_{n-1}^{(\alpha,\beta)}f(n-1).$$

- We consider the measure $d\mu_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}\,dx$, with $x \in [-1,1]$.

A discrete Hardy inequality in the Jacobi setting

- For $\alpha, \beta > -1$, we consider the sequences

$$a_n^{(\alpha,\beta)} = \frac{2}{2n+2+\alpha+\beta}\sqrt{\frac{(n+1)(n+1+\alpha)(n+1+\beta)(n+1+\alpha+\beta)}{(2n+1+\alpha+\beta)(2n+3+\alpha+\beta)}}$$

and

$$b_n^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)},$$

and the operator

$$J^{(\alpha,\beta)}f(n) = a_n^{(\alpha,\beta)}f(n+1) + b_n^{(\alpha,\beta)}f(n) + a_{n-1}^{(\alpha,\beta)}f(n-1).$$

- We consider the measure $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta\,dx$, with $x \in [-1,1]$.
- The orthonormal polynomials in $L^2([-1,1], d\mu_{\alpha,\beta})$ are

$$p_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)\|P_n^{(\alpha,\beta)}\|^{-1}_{L^2([-1,1],\mu_{\alpha,\beta})},$$

where $P_n^{(\alpha,\beta)}$ is the standard Jacobi polynomial.

A discrete Hardy inequality in the Jacobi setting

A discrete Hardy inequality in the Jacobi setting

- It is known that $J^{(\alpha,\beta)}p_n^{(\alpha,\beta)}(x) = xp_n^{(\alpha,\beta)}(x)$.

A discrete Hardy inequality in the Jacobi setting

- It is known that $J^{(\alpha,\beta)}p_n^{(\alpha,\beta)}(x) = xp_n^{(\alpha,\beta)}(x)$.
- Taking the operator $\mathcal{J}^{(\alpha,\beta)} = J^{(\alpha,\beta)} - I$, we have

$$\mathcal{J}^{(\alpha,\beta)}p_n^{(\alpha,\beta)}(x) = -(1-x)p_n^{(\alpha,\beta)}(x).$$

A discrete Hardy inequality in the Jacobi setting

- It is known that $J^{(\alpha,\beta)}p_n^{(\alpha,\beta)}(x) = xp_n^{(\alpha,\beta)}(x)$.
- Taking the operator $\mathcal{J}^{(\alpha,\beta)} = J^{(\alpha,\beta)} - I$, we have

$$\mathcal{J}^{(\alpha,\beta)}p_n^{(\alpha,\beta)}(x) = -(1-x)p_n^{(\alpha,\beta)}(x).$$

- The heat semigroup associated with $\mathcal{J}^{(\alpha,\beta)}$ is given by

$$W_t^{(\alpha,\beta)}f(n) = \sum_{k=0}^{\infty} f(k)K_t^{(\alpha,\beta)}(k,n)$$

where

$$K_t^{(\alpha,\beta)}(k,n) = \int_{-1}^{1} e^{-(1-x)t}p_k^{(\alpha,\beta)}(x)p_n^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}.$$

A discrete Hardy inequality in the Jacobi setting

- It is known that $J^{(\alpha,\beta)}p_n^{(\alpha,\beta)}(x) = xp_n^{(\alpha,\beta)}(x)$.
- Taking the operator $\mathcal{J}^{(\alpha,\beta)} = J^{(\alpha,\beta)} - I$, we have

$$\mathcal{J}^{(\alpha,\beta)}p_n^{(\alpha,\beta)}(x) = -(1-x)p_n^{(\alpha,\beta)}(x).$$

- The heat semigroup associated with $\mathcal{J}^{(\alpha,\beta)}$ is given by

$$W_t^{(\alpha,\beta)}f(n) = \sum_{k=0}^{\infty} f(k)K_t^{(\alpha,\beta)}(k,n)$$

where

$$K_t^{(\alpha,\beta)}(k,n) = \int_{-1}^{1} e^{-(1-x)t}p_k^{(\alpha,\beta)}(x)p_n^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}.$$

- $W_t^{(\alpha,\beta)}$ is the solution of the heat equation

$$\begin{cases} \dfrac{\partial u(j,t)}{\partial t} = \mathcal{J}^{(\alpha,\beta)}u(j,t), \\ u(j,0) = f(j). \end{cases}$$

A discrete Hardy inequality in the Jacobi setting

A discrete Hardy inequality in the Jacobi setting

• For $0 < \sigma < 1$, we define the fractional powers of $\mathcal{J}^{(\alpha,\beta)}$ by using the subordination formula

$$(\mathcal{J}^{(\alpha,\beta)})^{\sigma} f(n) = \frac{1}{\Gamma(-\sigma)} \int_0^{\infty} \left(W_t^{(\alpha,\beta)} f(n) - f(n) \right) \frac{dt}{t^{1+\sigma}}.$$

A discrete Hardy inequality in the Jacobi setting

- For $0 < \sigma < 1$, we define the fractional powers of $\mathcal{J}^{(\alpha,\beta)}$ by using the subordination formula

$$(\mathcal{J}^{(\alpha,\beta)})^{\sigma} f(n) = \frac{1}{\Gamma(-\sigma)} \int_0^{\infty} (W_t^{(\alpha,\beta)} f(n) - f(n)) \frac{dt}{t^{1+\sigma}}.$$

- We consider the sequence

$$\omega_{\sigma}^{(\alpha,\beta)}(n) = \frac{\Gamma\left(n+1+\beta+\frac{\alpha-\sigma+1}{2}\right)}{\Gamma\left(n+1+\beta+\frac{\alpha+\sigma+1}{2}\right)} \frac{\Gamma\left(n+1+\frac{\alpha-\sigma-1}{2}\right)}{\Gamma\left(n+1+\frac{\alpha+\sigma-1}{2}\right)}.$$

A discrete Hardy inequality in the Jacobi setting

- For $0 < \sigma < 1$, we define the fractional powers of $\mathcal{J}^{(\alpha,\beta)}$ by using the subordination formula

$$(\mathcal{J}^{(\alpha,\beta)})^{\sigma} f(n) = \frac{1}{\Gamma(-\sigma)} \int_0^{\infty} (W_t^{(\alpha,\beta)} f(n) - f(n)) \frac{dt}{t^{1+\sigma}}.$$

- We consider the sequence

$$\omega_{\sigma}^{(\alpha,\beta)}(n) = \frac{\Gamma\left(n+1+\beta+\frac{\alpha-\sigma+1}{2}\right)}{\Gamma\left(n+1+\beta+\frac{\alpha+\sigma+1}{2}\right)} \frac{\Gamma\left(n+1+\frac{\alpha-\sigma-1}{2}\right)}{\Gamma\left(n+1+\frac{\alpha+\sigma-1}{2}\right)}.$$

- We say that (α,β) belongs to the set V if $\alpha \geq \beta$ and and

$$\begin{aligned}(\alpha+\beta+1)&(\alpha+\beta+4)^2(\alpha+\beta+6)\\ &\geq (\alpha-\beta)^2((\alpha+\beta+1)^2 - 7(\alpha+\beta+1) - 24).\end{aligned}$$

A discrete Hardy inequality in the Jacobi setting: the result

A discrete Hardy inequality in the Jacobi setting: the result

Theorem

Let $(\alpha,\beta)\in V$, $0<\sigma<1/2$, *and* $f\in c_{00}(\mathbb{N})$. *Then*

$$\sum_{k=0}^{\infty} f(k)(\mathcal{J}^{(\alpha,\beta)})^{\sigma} f(k) \geq 2^{\sigma}\left(\frac{\Gamma\left(\frac{\alpha+\sigma+1}{2}\right)}{\Gamma\left(\frac{\alpha-\sigma+1}{2}\right)}\right)^2 \sum_{k=0}^{\infty} (f(k))^2 \omega_{\sigma}^{(\alpha,\beta)}(k).$$

A discrete Hardy inequality in the Jacobi setting: the first ingredient

A discrete Hardy inequality in the Jacobi setting: the first ingredient

- We consider the sequence

$$b_{-\sigma}^{(\alpha,\beta)}(n) = \frac{\Gamma\left(\frac{\alpha-\sigma+1}{2}\right)}{\Gamma\left(\frac{\alpha+\sigma+1}{2}\right)} \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \frac{\Gamma\left(n+1+\frac{\alpha+\sigma+1}{2}\right)}{\Gamma\left(n+\beta+1+\frac{\alpha-\sigma+1}{2}\right)} \times 2^{(\alpha-\sigma+1)/2+\beta}\omega_{\sigma}^{(\alpha,\beta)}(n).$$

A discrete Hardy inequality in the Jacobi setting: the first ingredient

- We consider the sequence

$$b_{-\sigma}^{(\alpha,\beta)}(n) = \frac{\Gamma\left(\frac{\alpha-\sigma+1}{2}\right)}{\Gamma\left(\frac{\alpha+\sigma+1}{2}\right)} \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \frac{\Gamma\left(n+1+\frac{\alpha+\sigma+1}{2}\right)}{\Gamma\left(n+\beta+1+\frac{\alpha-\sigma+1}{2}\right)} \times 2^{(\alpha-\sigma+1)/2+\beta}\omega_{\sigma}^{(\alpha,\beta)}(n).$$

Lemma

Let $\alpha, \beta > -1$ *and* $0 < \sigma < \min\{1, \alpha+1\}$. *Then*

$$(\mathcal{J}^{(\alpha,\beta)})^{\sigma} b_{-\sigma}^{(\alpha,\beta)}(n) = b_{\sigma}^{(\alpha,\beta)}(n).$$

A discrete Hardy inequality in the Jacobi setting: the second ingredient

A discrete Hardy inequality in the Jacobi setting: the second ingredient

Lemma

Let $(\alpha,\beta)\in V$, $0<\sigma<1/2$, *and* $f\in c_{00}(\mathbb{N})$. *Then*

$$(\mathcal{J}^{(\alpha,\beta)})^{\sigma}f(n)=\sum_{\substack{k=0\\k\neq n}}^{\infty}(f(n)-f(k))J_{\sigma}^{(\alpha,\beta)}(k,n)+f(n)E_{\sigma}^{(\alpha,\beta)}(n),$$

where

$$E_{\sigma}^{(\alpha,\beta)}(n)=\int_0^{\infty}\left(W_t^{(\alpha,\beta)}1(n)-1\right)\frac{dt}{t^{1+\sigma}}$$

and the kernel

$$J_{\sigma}^{(\alpha,\beta)}(k,n)=-\int_{-1}^{1}(1-x)^{\sigma}p_k^{(\alpha,\beta)}(x)p_n^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x)$$

is positive and symmetric. Moreover,

$$J_{\sigma}^{(\alpha,\beta)}(k,n)\leq C|k-n|^{-(1+2\sigma)},\qquad k\neq n.$$

Thank you for your attention!

¡Muchas gracias por vuestra atención!