ABOUT THE NOETHER'S THEOREM FOR FRACTIONAL LAGRANGIAN SYSTEMS AND A GENERALIZATION OF THE CLASSICAL JOST METHOD OF PROOF

by

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Lagrangian $L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $L(t,x,w,v)$,

$$\mathcal{L}_{\alpha,[a,b]}(x) = \int_a^b L\left(t, x(t), {}_cD^{\alpha}_{a+}x(t)\right) dt.$$

fractional Euler-Lagrange equation:

$$D^{\alpha}_{b-}\left(\frac{\partial L}{\partial v}(t, x(t), {}_cD^{\alpha}_{a+}x(t)\right) + \frac{\partial L}{\partial x}(t, x(t), {}_cD^{\alpha}_{a+}x(t)) = 0,$$

one parameter group of diffeomorphisms with no transformation in time

$$\phi_s : \mathbb{R}^n \mapsto \mathbb{R}^n,\ s \in \mathbb{R}$$

$$\int_{t_a}^{t_b} L\left(t, x(t), {}_cD^{\alpha}_{a+}x(t)\right) dt = \int_{t_a}^{t_b} L\Big(t, \phi_s(x)(t), {}_cD^{\alpha}_{a+}(\phi_s(x))(t)\Big) dt,$$

where $[t_a, t_b] \in [a, b]$

$$\frac{\partial L}{\partial v}(\star) \cdot {}_cD^{\alpha}_{a+}\left(\frac{d}{ds}\left(\phi_s(x)\right)|_{s=0}\right) - D^{\alpha}_{b-}\left(\frac{\partial L}{\partial v}(\star)\right) \cdot \frac{d}{ds}\left(\phi_s(x)\right)|_{s=0} = 0,$$

where $(\star) = \left(t, x(t), {}_cD^{\alpha}_{a+}x(t)\right)$.

$$\alpha = 1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(t, x(t), \dot{x}(t)) \cdot \frac{d}{ds}\phi_s(x)|_{s=0}\right) = 0$$

What about trasnformations also changing time ?

projectable or *fiber-preserving*

$$\begin{array}{rlcl} \phi_s : & [a,b]\times\mathbb{R}^n & \longrightarrow & \mathbb{R}\times\mathbb{R}^n \\ & (t,x) & \longrightarrow & (\varphi_s^0(t),\varphi_s^1(x)), \end{array}$$

In the classical case :

$$I(\tau,(t,x),(w,v)) = \frac{\partial L}{\partial v}(t,x,v) \cdot \left.\frac{d\varphi_s^1(x)}{ds}\right|_{s=0} + \left(L(t,x,v) - v\frac{\partial L}{\partial v}(t,x,v)\right) \left.\frac{\varphi_s^0(x)}{ds}\right|_{s=0}.$$

Proof ?

J. Jost and X. Li-Jost. *Calculus of Variations*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

P. J. Olver, *Applications of Lie groups to differential equations*, 2d edition, Graduate Textes in Mathematics, Springer-Verlag, 1993.

Step 1- rewrite the invariance condition in order to have constant interval

$$\tau = \varphi_s^0(t),$$

$$\int_a^b L\left(t, x(t), \frac{dx(t)}{dt}\right) dt = \int_a^b L\left(\varphi_s^0(t), (\varphi_s^1 \circ x)(t), \frac{d}{dt}\left(\varphi_s^1 \circ x\right)(t) \frac{1}{\frac{d\varphi_s^0(t)}{dt}}\right) \frac{d\varphi_s^0(t)}{dt} dt.$$

Step 2 – introduce the extended Lagrangian

the *extended Lagrangian* defined by

$$\tilde{L}(\tau, (t, x), (w, v)) := L\left(t, x, \frac{v}{w}\right) \cdot w,$$

$$t(\tau) := \tau,$$

$$\tilde{L}(\tau, t(\tau), x(\tau), \dot{t}(\tau), \dot{x}(\tau)) = L(\tau, x(\tau), \dot{x}(\tau)).$$

$$\int_a^b \tilde{L}(\tau, t(\tau), x(\tau), \dot{t}(\tau), \dot{x}(\tau)) d\tau = \int_a^b L\left(\tau, \phi_s(t(\tau), x(\tau)), \frac{d}{dt}\left(\phi_s(t(\tau), x(\tau))\right)\right) d\tau.$$

Apply the Noether's theorem on the case of transformations without changing time

Step 3- do not forget to verify that the solutions of the Euler-Lagrange Equation for L are solutions for the extended Lagrangian

$$\frac{d}{d\tau}\left[\frac{\partial L}{\partial v}(\star_\tau)\right] = \frac{\partial L}{\partial x}(\star_\tau)$$

$$\frac{d}{d\tau}L(\star_\tau) = \frac{\partial L}{\partial t}(\star_\tau) + \frac{d}{d\tau}\left(\frac{dx(\tau)}{d\tau}\frac{\partial L}{\partial v}(\star_\tau)\right),$$

Frederico G.S.F., Torres D.F.M., *A formulation of Noether's theorem for fractional problems of the calculus of variations*, J. Math. Anal. Appl. 334 (2007) 834-846.

Frederico G.S.F., Torres D.F.M., *Fractional Noether's theorem in the Riesz-Caputo sense*, Applied Mathematics and Computation, Volume 217, Issue 3, 1 October 2010, Pages 1023-1033.

R. A. C. Ferreira, A. B. Malinowska, *A counterexample to Frederico and Torres's fractional Noether-type theorem*, J. Math. Anal. Appl., Volume 429, Issue 2, 15 September 2015, Pages 1370-1373.

Why the proof presented by Frederico and Torres is wrong ?

Step 0 – variational symmetries

Definition 3 (**Variational symmetries**). — *The local group of transformation ϕ_s is a variational symmetry group of the functional (8) if whenever $I = [t_a, t_b]$ is a subinterval of $[a,b]$ and x is a smooth function defined over I such that its transform under ϕ_s denoted by $\tilde{x}$ is defined over $\tilde{I} = [\mu_a, \mu_b]$ which is a subset of $\phi_s^0([a,b]) = [\tau_a, \tau_b]$, then*

$$\mathcal{L}_{\alpha,a,I}(x) = \mathcal{L}_{\alpha,\tau(a),\tilde{I}}(\tilde{x}). \tag{20}$$

$$\int_{t_a}^{t_b} L\left(t, x(t), {}_cD^{\alpha}_{a+}x(t)\right) dt = \int_{\mu_a}^{\mu_b} L\left(\tau, \varphi_s^1 \circ x \circ (\varphi_s^0)^{-1}(\tau), {}_cD^{\alpha}_{\tau_a+}\left(\varphi_s^1 \circ x \circ (\varphi_s^0)^{-1}(\tau)\right)\right) d\tau.$$

Frederico-Torres definition :

$$\int_{t_a}^{t_b} L\left(t, x(t), {}_cD^{\alpha}_{a+}x(t)\right) dt = \int_{\tau_a}^{\tau_b} L\left(\tau, \varphi_s^1 \circ x \circ (\varphi_s^0)^{-1}(\tau), {}_cD^{\alpha}_{a+}\left(\varphi_s^1 \circ x \circ (\varphi_s^0)^{-1}(\tau)\right)\right) d\tau.$$

Localisation condition : $\phi_s^0(a) = a$ for all $s \in \mathbb{R}$.

Noether’s theorem for fractional Lagrangian systems

$$\frac{d\phi_s^0}{dt} = K(s), \qquad K(0) = 1$$

$$X = \zeta(t)\frac{\partial}{\partial t} + \xi(x)\frac{\partial}{\partial x},$$

$$I(x) = L(\star)\cdot\zeta + \int_a^t \Big[- [\partial_x L(\star)] \,.\, (\dot{x}\zeta - \xi) - \partial_v L(\star) . \Big(\zeta \cdot D^{\alpha}_{a+}[\dot{x}] + \dot{\zeta} \cdot D^{\alpha}_{a+}[x] - D^{\alpha}_{a+}(\xi) \Big) \Big]\, dt,$$

Step 1 – extended lagrangian

$${}_cD^{\alpha}_{\tau_a+}(\varphi^1_s \circ x \circ (\varphi^0_s)^{-1})(\tau)$$?

***Definition 4* (Admissible groups).** — *A local group of transformations* $\{\phi_s = (\phi^0_s, \phi^1_s)\}_{s\in\mathbb{R}}$ *is said to be admissible, if for all* $s \in \mathbb{R}$, ϕ^0_s *is an affine function of* t *of the form*

(51) $$\phi^0_s(t) = e^{\lambda s}t + \beta(s),$$

with $\beta(s)$ *satisfying* $\beta(s+s') = e^{\lambda s}\beta(s') + \beta(s)$ *for all* $s, s' \in \mathbb{R}$ *and* $\beta(0) = 0$.

$${}_cD^{\alpha}_{\phi^0_s(a)+}(y \circ (\phi^0_s)^{-1})(\tau) = {}_cD^{\alpha}_{a+}(y)(t)\frac{1}{\left(\dfrac{d\phi^0_s}{dt}\right)^{\alpha}}.$$

$$\int_{t_a}^{t_b} L\left(t, x(t), {}_cD_{a+}^{\alpha} x(t)\right) dt = \int_{t_a}^{t_b} L\left(\varphi_s^0(t), \varphi_s^1 \circ x(t), {}_cD_{a+}^{\alpha}(\varphi_s^1 \circ x)(t) \frac{1}{\left(\frac{d\varphi_s^0}{dt}\right)^{\alpha}}\right) \frac{d\varphi_s^0(t)}{dt} dt.$$

$$\tilde{L}_{\alpha}(\tau, (t, x), (w, v)) := L\left(t, x, \frac{v}{w^{\alpha}}\right) \cdot w.$$

$$\begin{aligned}\tilde{\mathcal{L}}_{\alpha,[a,b]}(t, x) &= \int_a^b \tilde{L}\left(t(\tau), x(t(\tau)), \frac{dt(\tau)}{d\tau}, {}_cD_{a+}^{\alpha} x(t(\tau))\right) d\tau \\ &= \int_a^b \tilde{L}(t, x, w, v) d\tau.\end{aligned}$$

Step 2 - invariance

$$\tilde{\mathcal{L}}_{\alpha,[a,b]}(t,x) = \tilde{\mathcal{L}}_{\alpha,[a,b]}(\phi_s(t,x)) = \tilde{\mathcal{L}}_{\alpha,[a,b]}(\varphi_s^0(t), \varphi_s^1(x)),$$

Invariance criterion

Lemma 9 (**Infinitesimal invariance criterion**). — *If the Lagrangian function $\tilde{\mathcal{L}}$ is invariant under the one parameter group $\{\phi_s = (\phi_s^0, \phi_s^1)\}_{s\in\mathbb{R}}$ then we have*

$$\partial_t \tilde{L}.\frac{d\phi_s^0}{ds}\mid_{s=0} + \partial_x \tilde{L}.\frac{d\phi_s^1}{ds}\mid_{s=0} + \partial_w \tilde{L}.\frac{d}{dt}\left(\frac{d\phi_s^0}{ds}\mid_{s=0}\right) + \partial_v \tilde{L}.D_{a+}^{\alpha}\left(\frac{d\phi_s^1}{ds}\mid_{s=0}\right) = 0 \quad (68)$$

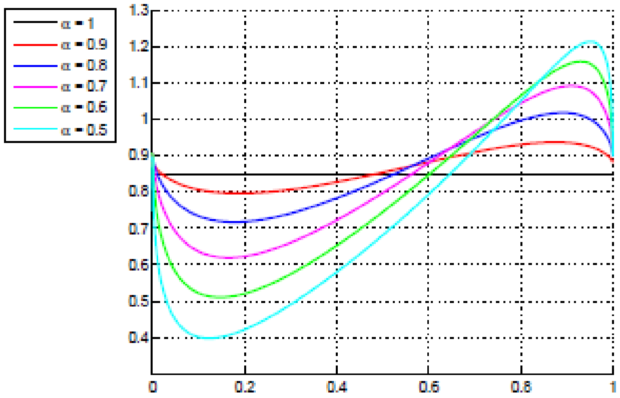
Step 3 – check that the solution of EL are solutions of the Extended EL

Not true in this case !

$$(CE_\alpha) \qquad \frac{\partial L}{\partial t}(\star_\tau) - \frac{d}{d\tau}\left(L(\star_\tau) - {}_cD^\alpha_{a+} x(\tau) \cdot \frac{\partial L}{\partial v}(\star_\tau) \right) = 0,$$

$$\begin{array}{llll} L: & [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (t, x, v) & \longrightarrow & \frac{1}{2}\left(\|x\|^2 + \|v\|^2 \right), \end{array}$$

$$Q_\alpha(x) := \frac{1}{2}\left(\|x\|^2 - \|v\|^2 \right) = \text{const.}$$



Lemma 10. — *Suppose* $G = \{\phi_s\}_{s\in\mathbb{R}}$ *is a one parameter group of symmetries of the variational problem* $\mathcal{L}_{\alpha,[a,b]}(x) = \int_a^b L\left(t, x(t), {}_cD^{\alpha}_{a+}x(t)\right) dt$ *satisfying the chain rule property. Let*

$$X = \zeta(t)\frac{\partial}{\partial t} + \xi(x)\frac{\partial}{\partial x}, \tag{69}$$

be the infinitesimal generator of G*. Then, we have:*

$$\partial_t L.\zeta + \partial_x.\xi + L.\dot{\zeta} + \partial_v L.\left(-D^{\alpha}_{a+}x.\dot{\zeta} + D^{\alpha}_{a+}(\xi)\right) = 0. \tag{70}$$

$$\frac{d}{dt}\left(L(\star)\right) = \partial_t L(\star) + \partial_x L(\star).\dot{x} + \partial_v L(\star).D^{\alpha}_{a+}[\dot{x}],$$

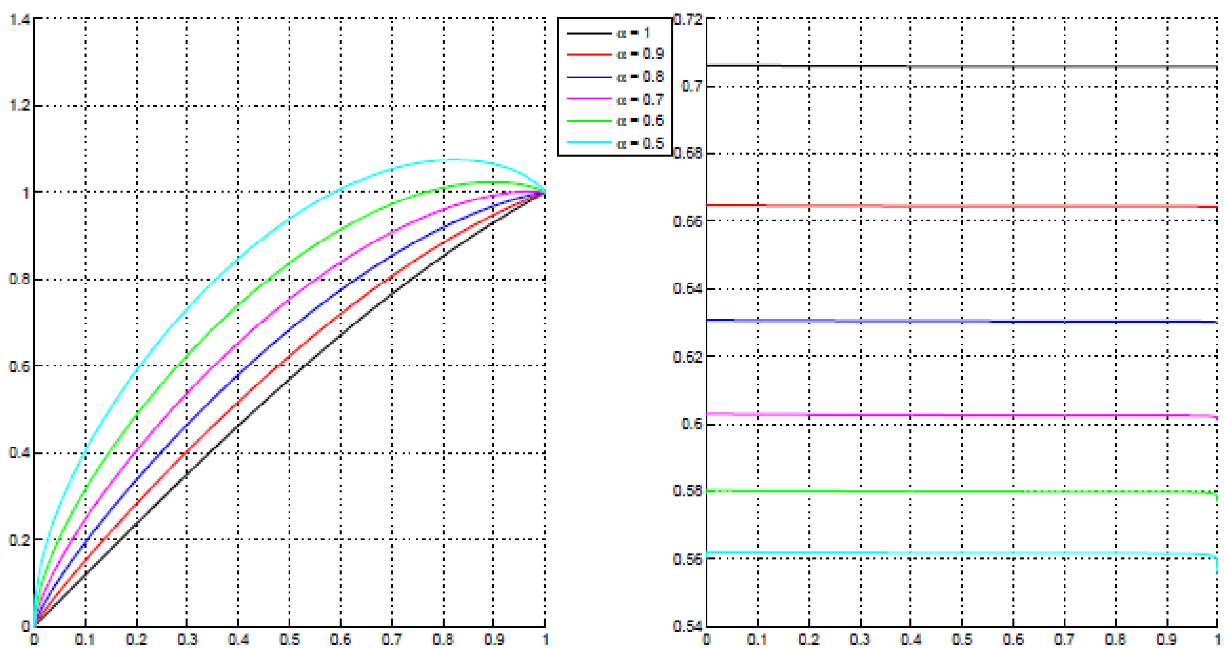
$$I(x) = L(\star)\cdot\zeta + \int_a^t \left[D^{\alpha}_{b-}\left[\partial_v L(\star)\right].\left(\dot{x}\zeta - \xi\right) - \partial_v L(\star).\left(\zeta\cdot D^{\alpha}_{a+}[\dot{x}] + +\dot{\zeta}\cdot D^{\alpha}_{a+}[x] - D^{\alpha}_{a+}(\xi)\right)\right] dt.$$

T. T. Atanackovic, S. Konjik, S. Pilipovic, S. Simic, *Variational problems with fractional derivatives: invariance conditions and Nöther's theorem*, Nonlinear Analysis 71 (2009) 1504-1517.

$$L = \frac{1}{2}\left({}_0D_t^\alpha u\right)^2 - \omega^2 \frac{1}{2}u^2,$$

$u(0) = 0$ and $u'(0) = 1$.

$$\frac{1}{2}\left({}_0D_t^\alpha u\right)^2 - \omega^2\frac{1}{2}u^2 + \int_0^t \left(-{}_0D_s^\alpha u' \cdot {}_0D_s^\alpha u + u' \cdot {}_sD_1^\alpha \left({}_0D_s^\alpha u\right)\right) ds = \text{const.}$$



The problem just identified….also in other proofs in different contex…
time scale, other fractional calculus, etc

Z. Bartosiewicz and D.F.M. Torres. Noether's theorem on time scales. *Journal of Mathematical Analysis and Applications*, 342(2):1220–1226, 2008.