Nonlinear Fractional Equations: Optimal Hölder Regularity.
Applications to a Posteriori Error Estimation

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Outline

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General description and motivation

Linear fractional problem

Nonlinear fractional problem

Fully nonlinear abstract evolution initial value problem

Let us consider the fully nonlinear abstract initial value problem

$$u'(t) = \mathcal{F}(u(t)), \quad 0 \le t \le T, \qquad \text{with} \qquad u(0) = u_0 \in \mathcal{B}, \qquad (1)$$

where,

- $\mathcal{F} : \mathcal{B} := \mathcal{D}(F) \subset Y \to X$ stands for certain nonlinear functional.
- X and Y are complex Banach spaces, $Y \subset X$ densely embeded.

The existence and uniqueness of solutions for (1) in the framework of Banach spaces is typically/frequently studied by means of

Hölder optimal regularity technics.

[1] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, 1995.

Functional setting: Hölder and weighted Hölder spaces

Recall that, given a Banach space $(Y, \|\cdot\|_Y)$:

- For $0 < \alpha < 1$, the set of α–Hölder continuous functions is defined as

$$C^\alpha([0,T];Y) := \{g : [0,T] \to Y : \|g\|_{C^\alpha([0,T];Y)} < +\infty\},$$

embedded by the norm

$$\|g\|_{C^\alpha([0,T];Y)} := \sup_{0 \le t \le T} \|g(t)\|_Y + \sup_{0 \le s < t \le T} \frac{\|g(t) - g(s)\|_Y}{(t-s)^\alpha}.$$

- For $0 < \alpha \le \gamma < 1$, the set of of weighted α–Hölder continuous functions is defined as

$$C^\alpha_\gamma((0,T];Y) := \{g : (0,T] \to Y : \|g\|_{C^\alpha_\gamma((0,T];Y)} < +\infty\},$$

embedded by the norm

$$\|g\|_{C^\alpha_\gamma((0,T];Y)} := \sup_{0 < t \le T} \|g(t)\|_Y + \sup_{0 \le s < t \le T} \frac{s^\gamma \|g(t) - g(s)\|_Y}{(t-s)^\alpha}.$$

Our contribution

We intend to extend the very well known results of [1] for the classical nonlinear problem (1) to the fractional (in time) version, in fact we look for the Hölder optimal regular solution of

$$\partial_t^{\beta} u(t) = \mathcal{F}(u(t)), \quad 0 \le t \le T, \qquad \text{with} \qquad u(0) = u_0 \in X, \qquad (2)$$

where ∂_t^{β} (hereafter solely ∂^{β}) stands for the fractional derivative in time of order β, with $1 < \beta < 2$, in the sense of Riemann–Liouville.

Notice that,

- Other definitions of fractional derivative could be considered, but not relevant differences will be found.
- By practical reasons, instead of (2) a semi–linear version have been considered (discussed latter).

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[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, 2006.
[3] J. Prüss, Evolutionary Integral Equations and Applications, 2012.

Semi–linear problem

Assume that $\mathcal{F}$ is Frêchet differentiable around a state $u^* \in \mathcal{B}$ (let say $u^* = u_0$), i.e. there exists the linear operator $\mathcal{F}_u(u_0)$ denoted by

$$A = \mathcal{F}_u(u_0).$$

Define the nonlinear functional $F : \mathcal{B} \subset Y \to X$ by

$$F(u) = \mathcal{F}(u) - Au.$$

Assume that $A : D(A) \subset X \to X$ is a sectorial operator.

Recall: There exist $a \in \mathbb{R}$, ($a = 0$ in our work) $M \geq 0$, and $0 < \theta < \pi/2$ such that his resolvent is analytic outside the sector

$$a + S_\theta := \{a + z \in \mathbb{C} : |\arg(-z)| < \theta\},$$

and bounded by

$$\|(z - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|z - a|}, \quad z \notin a + S_\theta.$$

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Semi–linear problem II

Then we can write (1) as follows

$$u'(t) = Au(t) + F(u(t)), \quad 0 \leq t \leq T, \quad \text{with} \quad u(0) = u_0 \in \mathcal{B}. \tag{3}$$

Or equivalently in integral form

$$u(t) = u_0 + \int_0^t Au(s)\,ds + F(u(t)), \quad 0 \leq t \leq T, \quad u_0 \in \mathcal{B}. \tag{4}$$

(for simplicity of the notation we denote $\int_0^t F(u(s))\,ds$ by $F(u(t))$ again)

We are now in a position to state our problem!!

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Semi–linear fractional problem

Consider the integral initial value problem of fractional order in time,

$$u(t) = u_0 + \partial^{-\beta} A u(t) + F(u(t)), \quad 0 \le t \le T, \quad u_0 \in \mathcal{B}. \qquad (5)$$

(instead of the fully nonlinear one (2)) where

- $\partial^{-\beta}$ defines the fractional integral of order $\beta > 0$ in the sense of Riemann–Liouville. Recall,

  $$\partial^{-\beta} g(t) := \int_0^t k_\beta(t-s) g(s)\, ds, \quad \text{where} \quad k_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0,$$

  for any integrable $g : (0, +\infty) \to X$,
- And it is assumed that $1 < \beta < 2$.

Aim of this work: We look for the optimal regular solution (in the sense of the Hölder continuity) of (5).

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[4] E.C. and R. Ponce, Optimal Hölder regularity for abstract semi-linear fractional differential equations in Banach spaces and their application to a posteriori error estimation of its time discretization (submitted 2018).

Prototype example: Fractional Burger's equation

Among other different formulations there exist in the literature, one can adopt the following format,

$$u(x,t) = u_0(x) + \partial^{-\beta}\partial_x^2 u(x,s) + \underbrace{\partial_x(u^2(x,t))}_{F(u(x,t))}, \quad (x,t) \in \Omega \times [0,T],$$

where,

- ∂_x stands for the (classical) first order spatial derivative.
- $\Omega \subset \mathbb{R}$ denotes certain spatial domain.
- $\partial_x \left(u^2(x,\cdot)\right)$ plays the role of $F(u)$.
- And joint with (e.g.) homogeneous Newmann bound. cond.

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[5] S. Dhawan, et al., Contemporary review of techniques for the solution of nonlinear burgers equation, J. Compu. Sci. 3 (2012).
[6] B. Lombard, et al., A fractional Burgers equation arising in nonlinear acoustics: Theory and numerics, IFAC Proceedings Volumes 46 (2013).
[7] K. Saad and E. HF. Al–Sharif, Analytical study for time and time–space fractional Burgers equation, Adv. Differen. Equ. (2017).

Other motivating works

This work is motivated as well by the widespread use of nonlinear fractional equations of type (1.3) in the context of anomalous diffusion phenomenon.
In fact if $1 < \beta < 2$, then it is applied as a super–diffusive model of anomalous type e.g. in heterogeneous media diffusion or in wave propagation in viscoelastic materials, and where the nonlinearity $F(u)$ represents the reaction term.

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[8] P. de Carvalho–Neto and G. Planas, Mild solutions to the time fractional Navier–Stokes equations in $\mathbb{R}^n$, J. Differential Equations 259 (2015), 2948-2980.
[9] A. Hanyga, Wave propagation in media with singular memory, Math. Comput. Model. 34 (2001), 1399-1421.
[10] R. Hilfer (ed.), Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[11] F. Mainardi and P. Paradisi, Fractional diffusive waves, J. Comput. Acoustics 9 (2001), 1417-1436.
[12] J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado (edts.) (eds.), Advances in Fractional Calculus. Theoretical Developments and Applications in Physics and Engineering, Springer, 2007.

...

Other motivating works

But it is also noticeable the great attention attracted by these problems from the theoretical point of view.

[13] B. Baeumer, M. Kovács, and H. Sankaranarayanan, Fractional partial differential equations with boundary conditions, J. Differential Equations 264 (2018).
[14] K. Li, J. Peng, and J. Jia, Cauchy problems for fractional differential equations with Riemann–Liouville fractional derivatives, J. Functional Analysis 263 (2012), 476-510. .
[15] Y. Li and Y. Wang, The existence and asymptotic behavior of solutions to fractional stochastic evolution equations with infinite delay, J. Differential Equations 266 (2019), 3514-3558. .
[15] C. Lizama and M. Murillo-Arcila, Maximal regularity in L^p spaces for discrete time fractional shifted equations, J. Differential Equations 263 (2017), 3175-3196.
[16] R. Ponce, Hölder continuous solutions for fractional differential equations and maximal regularity, J. Differential Equations 255 (2013), 3284-3304.
[16] J. Prüss, Evolutionary Integral Equations and Applications, Modern Birkhaüser Classics, Birkhaüser, Basel, 2012.
[17] R. Wang, D. Chen, and T. Xiao, Abstract fractional Cauchy problems with almost sectorial operators, J. Differential Equations 252 (2012), 202-235.
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Hypotheses

To conclude this section let us mention the main hypotheses:

(H1) Lipschitz continuity: There exist $R = R(u_0) > 0$ and $L = L(u_0) > 0$ such that

$$\|F_u(u_2) - F_u(u_1)\|_{\mathcal{L}(\mathcal{B},X)} \leq L\|u_2 - u_1\|_Y,$$

for all $u_1, u_2 \in \mathcal{B}$ with $\|u_j - u_0\|_Y \leq R$, $j = 1, 2$.

(H2) Sectoriality: $A = F_u(u_0) : D(A) \subset Y \to X$ is θ-sectorial, for some $0 < \theta < \pi/2$.

(H3) Equivalence of norms: The graph norm of A is equivalent to the norm of Y, that is, there exists $\gamma = \gamma(u_0) > 0$ such that

$$\frac{1}{\gamma}\|y\|_Y \leq \|y\|_{D(A)} := \|y\|_X + \|Ay\|_X \leq \gamma\|y\|_Y.$$

Outline

Mild solution of the linear equation

Consider first the linear problem,

$$v(t) = v_0 + \partial^{-\beta} A v(t) + f(t), \quad 0 \le t \le T. \tag{6}$$

- (Under certain constraints on β and θ) There exists a family of evolution operators $\{E_\beta(t)\}_{t \ge 0} \subset \mathcal{L}(X)$, such that the mild solution can be written as

$$v(t) = E_\beta(t) v_0 + \int_0^t E_\beta(t-s) f(s)\, \mathrm{d}s, \qquad 0 \le t \le T.$$

- In fact, the Laplace transform inversion formula allows to write

$$E_\beta(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{zt} z^{\beta-1} (z^\beta - A)^{-1}\, \mathrm{d}z, \qquad t \ge 0,$$

for a suitable complex path Γ connecting $-\mathrm{i}\infty$ and $+\mathrm{i}\infty$.

(the regularity of f in later stated!!)

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[18] E. Cuesta and C. Palencia, A numerical method for an integro–differential equation with memory in Banach spaces: Qualitative properties, SIAM J. Numer. Anal. 41 (2003).

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Important technical remark

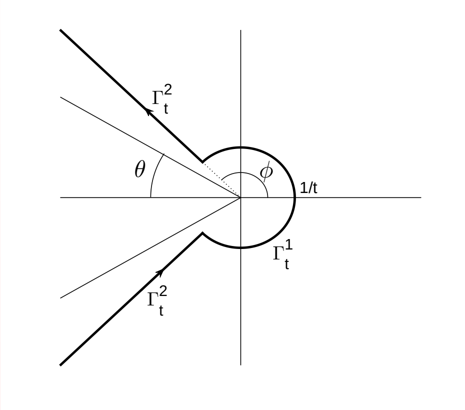

Figure: Complex path Γ_t.

Main Theorem

Let $1 < \beta < 2$, $v_0 \in D(A^{1+\varepsilon})$, and $f \in C^{\alpha}_{\gamma}((0,T];X^{\vartheta})$, with

- $0 < \dfrac{1-\gamma}{\beta} \le \varepsilon$.
- $0 < \dfrac{\beta-1}{\beta} \le \vartheta < 1$.
- $0 < \alpha \le \gamma < \alpha + \beta(\vartheta - 1) + 1$.

If v is the solution of (6), then there exists a (computable) constant $K > 0$ such that

$$\|v\|_{C^{\alpha}_{\gamma}((0,T];D(A))} \le K \left(\|v_0\|_{1+\varepsilon} + \|f\|_{C^{\alpha}_{\gamma}((0,T];X^{\vartheta})} \right).$$

The regularity assumed is oriented to the non–linear problem!

We denote by X^{ϑ} the domain of the fractional power $\vartheta > 0$ of A, that is $X^{\vartheta} := D(A^{\vartheta})$ endowed with the graph norm $\|x\|_{\vartheta} = \|x\| + \|A^{\vartheta}x\|$. In particular X^1 corresponds to $D(A)$, and X^0 to the space X.

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[1] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, 1995.

Key ingredients of the proof

Without getting lost in details, the proof is based on the following estimates:

Technical estimates: Let $\mu \geq 0$. Then the following estimates hold

$$\int_{\Gamma_t} \left| \frac{e^{zt}}{z^{\mu}} \right| |\,dz| \leq \left(C_\beta + \frac{2\, e^{\cos(\phi/\beta)}}{-\cos(\phi/\beta)} \right) t^{\mu-1},$$

and

$$\int_{\Gamma_t} \left| e^{zt} z^{\mu} \right| |\,dz| \leq \left(C_\beta + \frac{2\Gamma(\mu+1)}{(-\cos(\phi/\beta))^{\mu+1}} \right) \frac{1}{t^{\mu+1}},$$

for $t > 0$, where

$$C_\beta := \frac{1}{\beta} \int_{-\phi/\beta}^{\phi/\beta} e^{\cos(\psi/\beta)}\, d\psi.$$

Key ingredients of the proof

Operational estimates: Let x be such that $x \in D(A^{\vartheta})$. Then, for $t > 0$,

If $0 \leq \vartheta \leq 1$, then

$$\|E_{\beta}(t)x\| \leq \frac{1}{2\pi}\left(C_{\beta} + \frac{2\,e^{\cos(\phi/\beta)}}{-\cos(\phi/\beta)}\right)\left(\|x\| + \kappa\,(M+1)^{1-\vartheta}\|A^{\vartheta}x\|t^{\vartheta\beta}\right),$$

and

$$\|AE_{\beta}(t)x\| \leq \frac{\kappa(M+1)^{1-\vartheta}}{2\pi}\|A^{\vartheta}x\|\left(C_{\beta} + \frac{2\Gamma(\beta(1-\vartheta))}{(-\cos(\phi/\beta))^{\beta(1-\vartheta)}}\right)t^{\beta(\vartheta-1)}.$$

Moreover, if $\vartheta > \dfrac{\beta-1}{\beta}$, then

$$\left\|\int_0^t AE_{\beta}(s)x\,ds\right\| \quad \leq \quad \frac{\kappa\,(M+1)^{1-\vartheta}}{2\pi}\frac{\|A^{\vartheta}x\|}{\beta(\vartheta-1)+1}\cdot$$
$$\left(C_{\beta} + \frac{2\Gamma(\beta(1-\vartheta))}{(-\cos(\phi/\beta))^{\beta(1-\vartheta)}}\right)t^{\beta(\vartheta-1)+1}.$$

Note that...

Note that: All constant involved are in fact computable, in particular the constant K involved in the main Theorem which *collects* the computable constants involved in previous estimates.

Outline

Nonlinear fractional problem

We look for the optimal regularity (in the sense of Hölder continuity) for nonlinear fractional initial value problems

$$u(t) = u_0 + \partial^{-\beta} A u(t) + F(u(t)), \qquad 0 < t \leq T, \tag{7}$$

with $1 < \beta < 2$,

...but in a particular context...

...the context of time numerical discretizations!

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Time discretization

Let $\{U_n\}_{n=1}^N$ be a time discretization of (7) at time levels $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$, where

$$U_n \approx u(t_n) \qquad 0 \leq n \leq N,$$

$$I_n = [t_{n-1}, t_n], \qquad 1 \leq n \leq N,$$

and

$$\tau_n := t_n - t_{n-1}, \qquad 1 \leq n \leq N.$$

Without loss of generality we assume that the numerical method that provides the numerical solution above admits the format

$$U_n = U_0 + \sum_{j=0}^{n} q_{n-j} A U_j + \tau_n F(U_n), \qquad 1 \leq n \leq N, \tag{8}$$

for certain $\{q_n\}_{n=0}^N$ where each q_n depends in some manner on τ_n.

A lot of discretizations in the literature!!

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Assumptions

- The largest time step $\tau_{max} := \max_{0 \leq n \leq N}\{\tau_n\}$ is small enough (Typical and not relevant for our purposes).
- The nonlinear structure and the nonlocal nature of (7) and (8) make expected some regularity on U_n, in fact

$$\{U_n\}_{n=0}^{N} \subset X^{1+\vartheta}, \qquad \text{with} \qquad \frac{\beta - 1}{\beta} \leq \vartheta < 1,$$

- Since the estimates we provide below take into account the contribution of the initial error $e_0 := U_0 - u_0$, special attention must be paid to the regularity of the initial error e_0 rather than of U_0. In particular, we assume that for certain $0 < \varepsilon < 1$, (to be determined), the initial error satifies

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Some comments

- The order of convergence of the numerical method doesn't matter in this work, so the numerical scheme chosen doesn't deserve further attention.
- If $\beta = 1$, ϑ can be 0 and therefore spatial regularity is less demanding. This is consistent with the results achieved in [19] where it is merely required that $\{U_n\}_{n=0}^{N} \subset \mathcal{B}$.
- On the contrary to what we did in [19], here we assume that the initial error does not necessarily vanishes. This forces to some regularity on it.
- We have not gone yet in the practical implementation, but we guess that the assumptions (the ones above and below) can be relaxed in practical instances!

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[19] E. Cuesta and Ch. Makridakis, A posteriori error estimates and maximal regularity for approximations of fully non–linear parabolic problems in Banach spaces, Numer. Math. 111 (2008).

Continuous reconstruction

Define the continuous piecewise polynomial function

$$\mathcal{U} : [0, T] \to X^{1+\vartheta}, \qquad \mathcal{U} \in \mathcal{C}^1((0, T), X^{1+\vartheta}),$$

satisfying, for $1 \leq n \leq N$,

- $\mathcal{U}|_{I_n} \in \mathbb{P}_3(I_n, X^{1+\vartheta})$.
- $\mathcal{U}(t_n) = U_n$.
- $\mathcal{U}'|_{I_n}(t_n) = \mathcal{U}'|_{I_{n+1}}(t_n)$, for $1 \leq n \leq N - 1$.
- $\mathcal{U}'(0) = \mathcal{U}'(T) = 0$.

Here $\mathbb{P}_3(I_n, X^{1+\vartheta})$ denotes the set of all $X^{1+\vartheta}$–valued polynomials of degree less or equal to 3 defined in I_n.

Continuous formulation of the discrete scheme

Let $e : [0, T] \to \mathcal{B}$ be the error function $e(t) := \mathcal{U}(t) - u(t)$.

Let $\mathcal{R} : [0, T] \to \mathcal{B}$ be the (computable) residual function such that e is the solution of

$$e(t) = e_0 + \partial^{-\beta} Ae(t) + \overbrace{G(t, e(t)) + \mathcal{R}(t)}^{\text{original } F(e(t))}, \qquad 0 \le t \le T, \qquad (9)$$

where $G : [0, T] \times \mathcal{B} \to X$ is the function defined by

$$G(t, w) := F(\mathcal{U}(t)) - F(\mathcal{U}(t) - w), \quad 0 \le t \le T.$$

Note that,

- $G(t, e(t)) := F(\mathcal{U}(t)) - F(\mathcal{U}(t) - e(t)) = F(\mathcal{U}(t)) - F(u(t))$.
- $\mathcal{R}$ is in fact computable since it can be expressed in term of computable quantities as

  $$\mathcal{R}(t) = \mathcal{U}(t) - \mathcal{U}_0 - \partial^{-\beta} A\mathcal{U}(t) - F(\mathcal{U}(t)), \qquad 0 \le t \le T.$$

  This leads to the following equality holds

  $$\mathcal{U}(t) = \mathcal{U}_0 + \partial^{-\beta} A\mathcal{U}(t) + F(\mathcal{U}(t)) + \mathcal{R}(t), \qquad 0 \le t \le T.$$

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Additional hypothesis

The proof on the main result in this paper is based on the application of a fixed point theorem for certain functional

$$\Psi : \mathcal{Y}_\rho \to \mathcal{Y}_\rho$$

where,

$$\mathcal{Y}_\rho := \left\{ w \in C^\alpha_\gamma((0,T];D(A)) : w(0) = e_0, \text{ and } \|w\|_{C^\alpha_\gamma((0,T];D(A))} < \rho \right\}.$$

with

$$\rho \leq \frac{1}{2} R(u_0) < 1 \quad (R(u_0) \text{ comes out in (H1)}).$$

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Main Theorem

Assume that $u, \mathcal{U} : [0, T] \to \mathcal{B}$, the continuous and discrete solutions respectively satisfy (among the initial Hypotheses (H1)–(H3))

- The constants $\alpha, \beta, \gamma, \varepsilon$, and ϑ hold the hypotheses stated in the main result for the linear case.
- The constant $\rho > 0$ holds

$$\rho < \frac{1}{6KL},$$

  where K is the constant obtained in main Theorem.

- There holds

$$\|e_0\|_{1+\varepsilon} + \|\mathcal{R}\|_{C^{\alpha}_{\gamma}((0,T];X^{\vartheta})} \leq \frac{\rho}{K}\left(1 - \frac{9KL\rho}{2}\right),$$

Then there exists a computable constant $C > 0$ such that

$$\|\underbrace{\mathcal{U} - u}_{e}\|_{C^{\alpha}_{\gamma}((0,T];D(A))} \leq C\left(\|e_0\|_{1+\varepsilon} + \|\mathcal{R}\|_{C^{\alpha}_{\gamma}((0,T];X^{\vartheta})}\right).$$

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Some ideas on the proof

- Under the same hypotheses as in the main Theorem, $G(\cdot, w(\cdot)) \in C^{\alpha}_{\gamma}((0,T];X^{\vartheta})$, for every $w \in \mathcal{Y}_{\rho}$, and there holds

$$\|G(\cdot, w(\cdot))\|_{C^{\alpha}_{\gamma}((0,T];X^{\vartheta})} \leq \frac{9L\rho}{2}\|w\|_{C^{\alpha}_{\gamma}((0,T];D(A))},$$

- We make use of a fixed point theorem applied to $\Psi : \mathcal{Y}_{\rho} \to \mathcal{Y}_{\rho}$ defined by $\Psi(w) = h$, for $w \in \mathcal{Y}_{\rho}$, where h is the solution of the linear equation

$$h(t) = e_0 + \partial^{-\beta}Ah(t) + G(t, w(t)) + \mathcal{R}(t), \quad 0 \leq t \leq T.$$

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Final comments

- We achieved true a posterior error estimations, i.e. all *ingredients* are actually computable.
- This stands for a first step, the theoretical part, leading to the practical implementation.
- The most disappointing is the nonlocal nature of the norms involved, and the tricky/fine regularity requirements.
- In spite of the previous comment, since the estimates (all bounds) we have been achieved are *optimal* in the sense of Hölder regularity, it is expected than in practical instances some requirements can be relaxed in order to get an easier implementation.
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So many thanks for your attention!!