An extension of a Theorem of Domar on invariant subspaces

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Joint work with Eva A. Gallardo-Gutiérrez (Madrid, Spain) and Jonathan R. Partington (Leeds, UK).

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Preliminares

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For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the Fourier Transform is defined by

$$(\mathcal{F}f)(x) = \widehat{f}(x) = \int_{-\infty}^{\infty} f(t) \exp(-ixt) dt, \quad (x \in \mathbb{R}).$$

Plancherel Theorem

$$f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \longrightarrow \widehat{f} \in L^2(\mathbb{R})$$

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Theorem

For every measurable set $E \subset \mathbb{R}$, the subspace $\mathcal{M}_E$ is a closed translation-invariant subspace of $L^2(\mathbb{R})$. Moreover, every closed translation-invariant subspace of $L^2(\mathbb{R})$ is $\mathcal{M}_E$ for some measurable set E and $\mathcal{M}_A = \mathcal{M}_B$ if and only if

$$m((A \setminus B) \cup (B \setminus A)) = 0.$$

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The clasical Paley-Wiener Theorem states that the Laplace Transform

$$(\mathcal{L}f)(s) = \int_0^\infty f(t) \exp(-st) dt, \quad (s \in \mathbb{C}_+),$$

induces (up to the constant) an unitary equivalence between $L^2(\mathbb{R}_+)$ and $H^2(\mathbb{C}_+)$.

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Note that $\{S_\tau\}_{\tau \geq 0}$ has a rich lattice of invariant subspaces.

Weighted L^2 spaces

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Bergman spaces $A^2_\alpha(\mathbb{C}_+)$ for $(\alpha > -1)$

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Zen spaces $A^2_\nu(\mathbb{C}_+)$

$A^2_\nu(\mathbb{C}_+) = \{F \in \mathcal{H}(\mathbb{C}_+) : \sup_{\varepsilon>0} \int_{\overline{\mathbb{C}_+}} |F(z+\varepsilon)|^2 d\nu(z) < \infty\}$, with $d\nu = d\tilde{\nu} \otimes dx$ where $\nu, \tilde{\nu}$ are both positive regular Borel measures on $\mathbb{C}_+$ and $\mathbb{R}_+$, resp., and $\tilde{\nu}$ satisfies a doubling condition.

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Question

How is the structure of $Lat\{S_\tau : \tau \geq 0\}$ in $L^2(\mathbb{R}_+, \omega(t)dt)$?

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For all $a \in \mathbb{R}_+ \cup \{0\} \cup \{\infty\}$, the *"standard subspaces"* are

$$L^2([a,\infty), \omega(t)dt) = \{f \in L^2(\mathbb{R}_+, \omega(t)dt) : f(t) = 0 \text{ a.e. } 0 \leq t \leq a\}.$$

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But, it is known is the existence of fairly large classes of weights for which the lattices have non-standard subspaces: classes characterized by bounds on the size of the weight function ω and $1/\omega$ at infinity (Atzmon, Borichev-Hedenlman, Domar, Nikolskii, Thomas...)

Domar's Theorem

Theorem (Y. Domar, 1983)

Let ω be a positive continuous function in $\mathbb{R}_+$ such that $\log \omega$ is concave in $[c, \infty)$ for some $c \geq 0$. Assume that

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In this case, observe that the semigroup $\{S_\tau : \tau \geq 0\}$ has very few closed invariant subspaces in $L^2(\mathbb{R}_+, \omega(t)dt)$.

About Domar's Theorem

- Transfer to a problem in complex function theory.
- Accurate bounds of Laplace Transform of the functions considered.
- Use the Ahlfors-Beurling Theorem on bounds of complex analytic functions in an n-connected domain.

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Domar's Problem 3

How relevant is the concavity assumption on $\log \omega$?.

Extension of Domar's Theorem

A First Approach

Recall that a sequence of positive real numbers $\{a_n\}_{n=1}^{\infty}$ is said to be *logarithmically concave* if $a_n^2 \geq a_{n-1}a_{n+1}$ for all $n \geq 2$.

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Key ingredient result

Proposition

Let $\omega > 0$ be a continuous decreasing function in $\mathbb{R}_+$. Let $\mathcal{M}$ be a non-trivial invariant subspace for $\{S_\tau\}_{\tau \geq 0}$ in $L^2(\mathbb{R}_+, \omega(t)dt)$. Assume $\mathcal{M}$ contains a non-trivial standard invariant subspace for $\{S_\tau\}_{\tau \geq 0}$. Then $\mathcal{M}$ is standard.

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Key idea: Construct a positive continuous function ω_a in $\mathbb{R}_+$ (related to ω) which satisfies the hypotheses of Domar's Theorem.

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For each $n \geq 0$, let Λ_n be the linear function defined in the interval $[n, n+1)$ by

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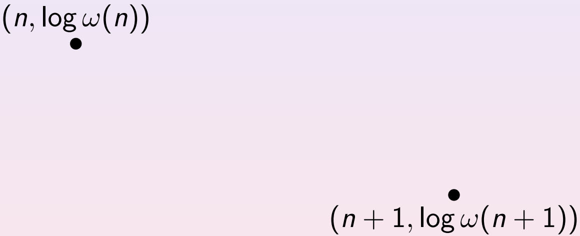

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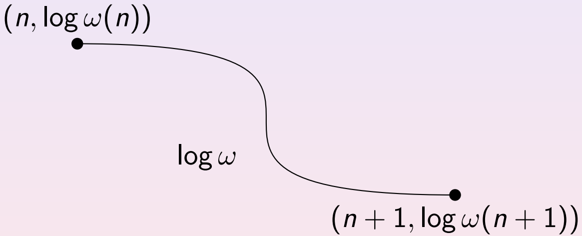

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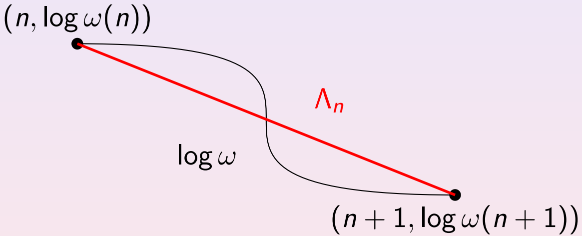

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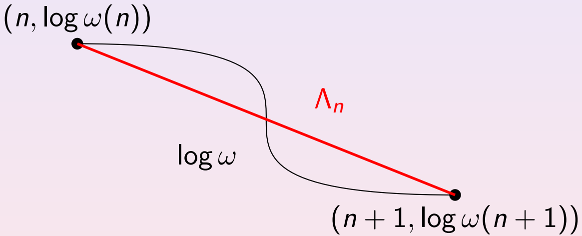

Consider the piecewise function

$$\Lambda(t) = \sum_{n \geq 0} \Lambda_n(t) \chi_{[n,n+1)}(t), \quad (t \in \mathbb{R}_+)$$

which is concave for $t \geq 1$.

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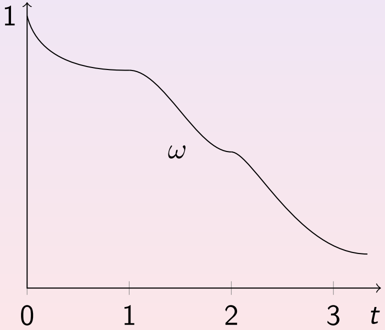

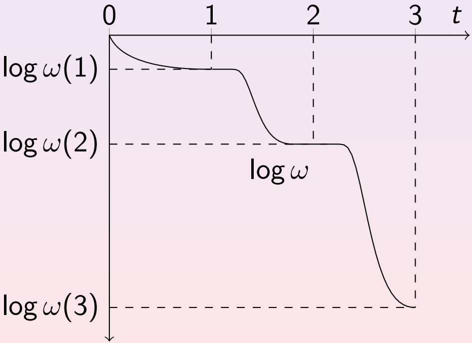

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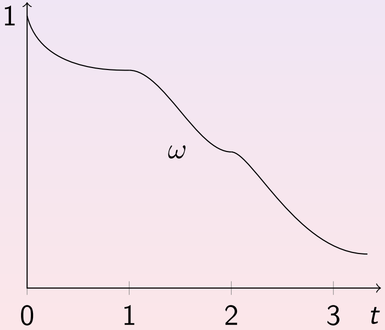

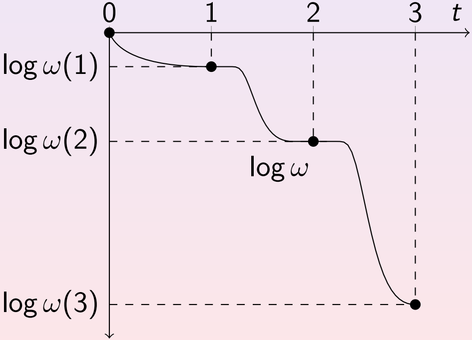

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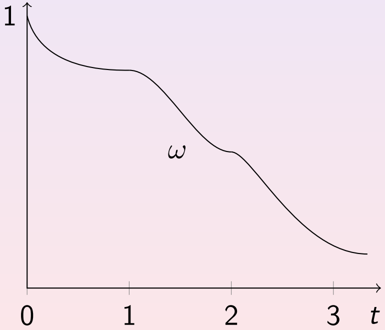

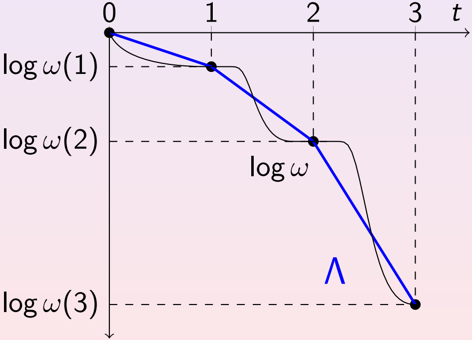

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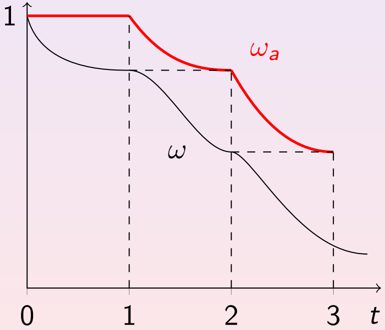

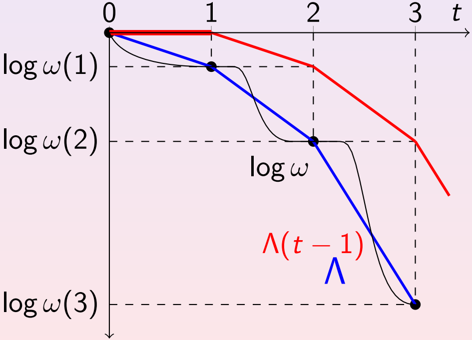

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Observe that $c_a < \infty$, since $\mathcal{M} \neq \{0\}$.

Using a density argument and the chain $\overline{S_2\mathcal{M}}^{\omega_a} \subset \overline{S_2\mathcal{M}}^{\omega} \subset \mathcal{M}$, it follows that

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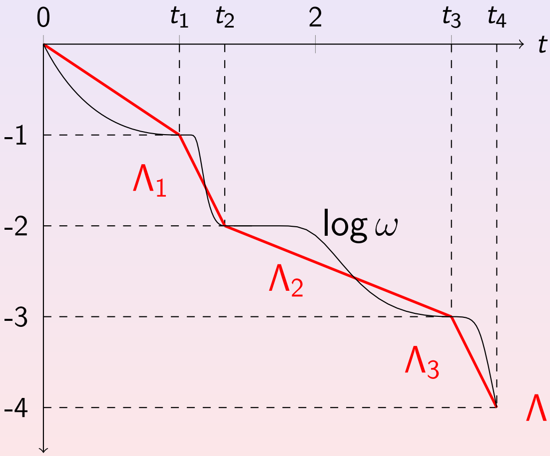

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Then all closed invariant subspaces for $\{S_\tau\}_{\tau \geq 0}$ in $L^2(\mathbb{R}_+, \omega(t)\,dt)$ are standard.

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is decreasing. Therefore, we recover the logarithmically concave condition (H1) of Theorem 1 whenever $\{t_n\}_{n=1}^{\infty}$ consists of equidistant points in $\mathbb{R}_+$.

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6. Hence, ω_a satisfies the required conditions and we continue as in the proof of the previous theorem.

Star-shaped weights

In the context of Banach algebras, well-behaved sequences of weights $\{\omega(n)\}_{n=0}^{\infty}$ have played an important role in order to ensure that all closed ideals in $\ell^1(\omega(n))$ are standard, where

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Observe that for $k \in \mathbb{N} \cup \{\infty\}, \ldots$, we have the *"standard ideals"*

$$\mathcal{M}_k = \left\{ \sum_{n=0}^{\infty} y_n z^n \in \ell^1(\omega(n)) : \ y_0 = y_1 = \cdots = y_{k-1} = 0 \right\}$$

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Thomas proved that for a star-shaped weight $\{\omega(n)\}_{n=0}^{\infty}$, all the closed ideals of $\ell^1(\omega(n))$ are standard whenever $\omega(n)^{1/n}$ is $O(1/n^a)$ for some $a > 0$.

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Thomas proved that for a star-shaped weight $\{\omega(n)\}_{n=0}^{\infty}$, all the closed ideals of $\ell^1(\omega(n))$ are standard whenever $\omega(n)^{1/n}$ is $O(1/n^a)$ for some $a > 0$.

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Proposition

Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}_+$ be a logarithmically concave non-zero sequence with $a_0 = 1$. Then the sequence $\{a_n^{1/n}\}_{n=1}^{\infty}$ decreases. Moreover, if $a_{n+1}/a_n \searrow 0$ as $n \to \infty$, then $\{a_n\}_{n=1}^{\infty}$ is star-shaped.

Note that not every star-shaped weight is logartimically concave.

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For instance, consider the positive decreasing sequence $\{a_n\}_{n=0}^{\infty}$ given by $a_0 = 1$ and

$$a_{2n-1} = \exp(-(2n-1)^2), \quad a_{2n} = \exp(-(2n)^2 + 2n - 1), \quad (n \geq 1).$$

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In the context of Theorem 2, is posible to find a strictly increasing sequence $\{t_n\}_{n=0}^{\infty} \subset \mathbb{R}_+$, with $t_n \to \infty$ as $n \to \infty$ such that

$$\{(w(t_{n+1})/w(t_n))^{1/(t_{n+1}-t_n)}\}_{n=0}^{\infty}$$

decreases but $\{w(t_n)\}_{n=0}^{\infty}$ is not star-shaped weight.

Thank you for your attention

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