On domains of functions of Cesàro operators. Poisson equation and discrete Hilbert transform

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For $T \in \mathcal{B}(X)$, put $\mathcal{T}(n) := T^n$ for $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, with $T^0 = I_X$.

For $\alpha \geq 0$ and $n \in \mathbb{N}_0$, set

$$\Delta^{-\alpha}\mathcal{T}(n) := (k^\alpha * \mathcal{T})(n) = \sum_{j=0}^{n} k^\alpha(n-j)T^j \quad \text{(Cesàro sum of order } \alpha\text{)}$$

and

$$M_T^\alpha(n) := \frac{1}{k^{\alpha+1}(n)}\Delta^{-\alpha}\mathcal{T}(n) \quad \text{(Cesàro mean of order } \alpha\text{)},$$

where $k^0 = \delta_0$ and, for $\alpha > 0$,

$$k^\alpha(n) := \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}, \; n \in \mathbb{N}; \; k^\alpha(0) = 1,$$

- T is (C,α)-bounded if : $\sup_n \|M_T^\alpha(n)\| < \infty$,

$(C,0)$-bounded = power-bounded: $\sup_{n\geq 0} \|T^n\| < \infty$

$(C,1)$-bounded = Cesàro mean-bounded: $\sup_n \frac{1}{n+1}\|\sum_{j=0}^{n} T^j\| < \infty$.

- $T \in \mathcal{B}(X)$ is (C,α)-ergodic if there exists $P_\alpha\colon X \to \mathrm{Ker}(I-T)$.

$$\begin{aligned} P_\alpha x : &= \lim_{n\to\infty} M_T^\alpha(n)x = \lim_{n\to\infty} \frac{1}{k^{\alpha+1}(n)} \Delta^{-\alpha}\mathcal{T}(n)x \\ &= \lim_{n\to\infty} \frac{1}{k^{\alpha+1}(n)} \sum_{j=0}^{n} k^\alpha(n-j)T^j x, \quad \forall x \in X. \end{aligned}$$

$(C,1)$-ergodic = mean-ergodic: $\quad P_1 x = \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} T^j x.$

- (C,α)-ergodic $\Longrightarrow$ (C,α)-bounded.

- Mean ergodic theorems (Von Neumann, Riesz, Kakutani, Eberlein, ...):

For T power-bounded, $x \in X$,

$$\text{There exists } P_1 x \Longleftrightarrow x \in \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T) \subseteq X$$

Hence, T is mean-ergodic if and only if $X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Ran}}(I-T)$.
(Example: X reflexive, Roch).

- What about $\alpha \neq 0$, $\alpha \neq 1$? Quite a number of papers on ergodicity of (C,α)-bounded operators, growth of Cesáro sums and Cesáro means of order α.

MOTIVATION

- Focus on specific line.
- Ergodicity $\longrightarrow$ (Markov chains $\mathcal{M}$, operators $\mathcal{P}_{\mathcal{M}}$) $\longleftarrow$ Probability theory:

To obtain central limit theorems for elements $y \in \mathrm{Ran}(\mathrm{I} - \mathcal{P}_{\mathcal{M}}) \subseteq \mathrm{X}$ $(X = L_p)$

For power-bounded operator $T \in \mathcal{B}(X)$, find $x \in X$ such that

$$P_1 x = \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} T^j x (= \lim_{n\to\infty} M_T^1(n)x)$$

with suitable convergence rate (polynomial, for instance).

- Problem related with an appropriate description of elements

$$y \in (I - T)^s X \subseteq X, \quad 0 < s \leq 1,$$

so that . . . fractional Poisson equation [s-PoEq]:

$$(I - T)^s x = y, \quad x, y \in X.$$

• Solution (Derriennic-Lin 2001):
T power-bounded and mean ergodic; given $y \in X$,

$$s - \text{[PoEq] solution } x \Longleftrightarrow \sum_{n=0}^{\infty} \frac{1}{n^{1-s}} T^n y \text{ converges,}$$

and then $x = \sum_{n=1}^{\infty} c_n(s) T^n y$ where $c_n(s) \sim n^{s-1}$ as $n \to \infty$.

Characterization $\Longrightarrow \| \frac{1}{n} \sum_{j=1}^{n} T^j y \| = o(\frac{1}{n^s})$ as $n \to \infty$.

• $((I - T)^s)_{\Re s > 0}$ is a (holomorphic) semigroup in $\mathcal{B}(X)$, and a C_0-semigroup provided $(I - T)X$ is dense in X. Let $\log(I - T)$ denote its infinitesimal generator.

• QUESTION: Whether or not $-\log(I - T)$ coincides with the one-sided ergodic Hilbert transform

$$\mathcal{H}_T := \sum_{n=1}^{\infty} \frac{T^n}{n}.$$

- $\mathcal{H}_T \subseteq \log(I - T)$ (Derriennic-Lin)
- $\mathcal{H}_T = -\log(I - T)$, for particular T, X (Assani-Lin 2007; Cohen-Lin 2009; Cuny-Lin 2010).

METHODOLOGY

- Cohen-Cuny-Lin and *Haase-Tomilov*, 2010, in full generality,

$$\mathcal{H}_T = -\log(I-T).$$

- Haase-Tomilov, conceptual approach. Recover Derriennic-Lin's on [PoEq].
  - *Functional calculus* (analytic functions and sequences):

$$A^0(\mathbb{D}) \equiv \ell^1 \to \mathcal{B}(X), \quad \mathfrak{f} \mapsto \mathfrak{f}(T) := \sum_{n=0}^{\infty} f(n) T^n$$

  - *Admissibility*: To extend FC to suitable, admissible, $\mathfrak{f}$ analytic in $\mathbb{D}$. Example, Taylor coefficients forming logarithmically convex sequences (Th. Kaluza, 1928). Apply to $\log(1-z)$ and $(1-z)^{-s}$.
  - *Unbounded operatorial functions*: $\operatorname{Dom} \mathfrak{f}(T)$, admissible $\mathfrak{f}$.

$$\operatorname{Ran}(I-T)^s \sim \operatorname{Dom}(I-T)^{-s}, \quad \text{for } I-T \text{ injective.}$$

$$\text{Also,} \quad \operatorname{Dom} \log(I-T), \quad \text{for } \mathcal{H}_T = -\log(I-T).$$

- So that

$$x \in \operatorname{Dom} \mathfrak{f}(T) \text{ if and only if } \sum_{n=1}^{\infty} f(n) T^n x \text{ is convergent.}$$

OBJECTIVES and TOOLS

- Previous results: Power-boundedness and mean ergodicity.

- Lot of results on boundedness, growth and ergodidity for (C, α)-bounded, integer and fractional $\alpha > 0$. Nothing on Poisson or Hilbert transform.

- AIM: To extend results on Poisson equation and
  Hilbert transform for (C, α)-bounded operators.

- Follow Haase-Tomilov. NEED:
  – Functional calculus: $f \in \tau^\alpha \subseteq \ell^1$, $A^\alpha(\mathbb{D}) := \{\mathfrak{f} \mid f \in \tau^\alpha\}$, such that

$$\|f\|_{(\alpha)} := \sum_{n=0}^{\infty} |W^\alpha f(n)| k^{\alpha+1}(n) < \infty,$$

  where $W^\alpha f$ is the Weyl difference of f order α.

  – Admissibility: α-admissibility; using $A^\alpha(\mathbb{D})$, difference operators.

  – Generalization of Kaluza's theorem: "Logarithmic convexity"for higher order differences. Ex.: $(1-z)^{-s}$, $\log(1-z)$.

- *Fractional differences.* For a sequence f and $n \in \mathbb{N}_0$ define

$$D^1 f(n) = W^1 f(n) := f(n) - f(n+1),$$

and subsequently

$$D^m f(n) = W^m f(n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(n+j), \quad m \in \mathbb{N}.$$

Then the Weyl difference of f of order $\alpha > 0$ is

$$W^\alpha f(n) = W^m W^{-(m-\alpha)} f(n), \quad m = [\alpha] + 1,$$

with inverse the Weyl sum of order $\alpha > 0$

$$W^{-\alpha} f(n) := \sum_{j=n}^{\infty} k^\alpha (j-n) f(j) = \sum_{j=0}^{n} \binom{n-j+\alpha-1}{\alpha-1} f(j),$$

- Set $D^\alpha f(n) := \sum_{j=n}^{\infty} k^{-\alpha}(j-n) f(j)$. *Sometimes,*

$$D^\alpha f = W^\alpha f \text{ or } W^{-\alpha}(D^\alpha f) = f.$$

- Define τ^α by: $f \in \tau^\alpha$ if and only if $f \in \ell^1$ and

$$\|f\|_{(\alpha)} := \sum_{n=0}^{\infty} |W^\alpha f(n)| k^{\alpha+1}(n) < \infty$$

- $W^\alpha \colon \tau^\alpha \to \ell^1(k^{\alpha+1})$ surjective isometry (inverse $W^{-\alpha}$); $W^\alpha = D^\alpha$ on τ^α.
- $\tau^\beta \hookrightarrow \tau^\alpha \hookrightarrow \ell^1$, for $\beta > \alpha > 0$, and $k^\beta \in \tau^\alpha$ if $\Re\beta < 0$ or $\beta = 0$, for all $\alpha \geq 0$.
- τ^α Banach algebra wrt $(f * g)(n) = \sum_{j=0}^{n} f(n-j)g(j)$. Gelfand transform:

$$f \in \tau^\alpha \to \mathfrak{f} \in A^\alpha(\mathbb{D}) := \{\mathfrak{f} \,:\, f \in \tau^\alpha\} \subset A^0(\mathbb{D}); \ \|\mathfrak{f}\|_{A^\alpha(\mathbb{D})} := \|f\|_{(\alpha)}$$

- Denote

$$\Delta^{-\alpha}\mathcal{Z}(n) := (k^\alpha * \{z^j\})(n) = \sum_{j=0}^{n} k^\alpha(n-j) z^j, \quad \alpha > 0.$$

Then $|\Delta^{-\alpha}\mathcal{Z}(n)| \leq k^{\alpha+1}(n)$,

$$\mathfrak{f}(z) = \sum_{n=0}^{\infty} W^\alpha f(n) \Delta^{-\alpha}\mathcal{Z}(n), \quad z \in \overline{\mathbb{D}}, \mathfrak{f} \in A^\alpha(\mathbb{D}).$$

FUNCTIONAL CALCULUS

- $T \in \mathcal{B}(X)$ is (C,α)-bounded if and only if there exists $\Theta_\alpha : \tau^\alpha \to \mathcal{B}(X)$,

$$\Theta_\alpha(f)x = \sum_{n=0}^{\infty} W^\alpha f(n) \Delta^{-\alpha} \mathcal{T}(n)x, \quad x \in X,\ f \in \tau^\alpha;$$

- *Functional calculus* $\Phi_\alpha \colon A^\alpha(\mathbb{D}) \to \mathcal{B}(X)$,

$$\mathfrak{f}(T) \equiv \Phi_\alpha(\mathfrak{f}) := \Theta_\alpha(f), \quad \mathfrak{f}(z) = \sum_{n=0}^{\infty} f(n) z^n.$$

- *α-regularization*, $\alpha > 0$: For T (C,α)-bounded, $\mathfrak{f}$ holomorphic in $\mathbb{D}$ is α-regularizable (wrt T) if there is $\mathfrak{e} \in A^\alpha(\mathbb{D})$ s.t. $\mathfrak{e}\mathfrak{f} \in A^\alpha(\mathbb{D})$, $\mathfrak{e}(T)$ injective.

  Then

$$\mathfrak{f}(T) := \mathfrak{e}(T)^{-1}(\mathfrak{e}\mathfrak{f})(T) \quad [[\text{ closed }]]$$

  (regularization for $\alpha = 0$ is wrt $\ell^1 \equiv A^0(\mathbb{D})$).

Example

Assume $\mathrm{Ker}(I-T)=\{0\}$, so $(I-T)$ is injective; $\mathfrak{e}\equiv 1-z$ α-regulariser.

(1) Let $s\in\mathbb{R}$ and $\mathfrak{k}^s\leftrightarrow k^s$. Take $n<s\le n+1$ with $n\in\mathbb{N}_0$ so

$$(1-z)^{n+1}\mathfrak{k}^s(z)=\mathfrak{k}^{s-n-1}(z)\in A^\alpha(\mathbb{D}).$$

Then

$$(I-T)^{-s}:=\mathfrak{k}^s(T)=(I-T)^{-n-1}\mathfrak{k}^{s-n-1}(T).$$

(2) Note

$$(1-z)\log(1-z)=-z+\sum_{n=1}^{\infty}\frac{z^{n+1}}{n(n+1)}\in A^\alpha(\mathbb{D}),$$

whence

$$\log(I-T):=(I-T)^{-1}\left[(1-z)\log(1-z)\right](T)$$

- One has $\overline{\mathrm{Ran}}(I-T)^s=\overline{\mathrm{Ran}}(I-T)$, $\log(I-T)$ = generator of $(I-T)^s$, $s>0$.

Definition

Let $\alpha \geq 0$ and let $\mathfrak{f}(z) = \sum_{n=0}^{\infty} f(n) z^n$ be a holomorphic function on $\mathbb{D}$. Then $\mathfrak{f}$ is said to be α-*admissible* if:

- (i) The sequence f is bounded, $D^{\beta} f(n) \geq 0$ and $W^{-\beta}(D^{\beta} f) = f$ for $\beta \in \{0, \alpha\}, n \in \mathbb{N}_0$.
- (ii) The function $\mathfrak{f}$ does not have zeros in $\mathbb{D}$ and if $g \longleftrightarrow \dfrac{1}{\mathfrak{f}}$ then

  $W^{\beta} g$ exist and satisfy , for $\beta \in \{0, \alpha\}$,

$$W^{\beta} g(0) \geq 0, \quad W^{\beta} g(n) \leq 0 \, (n \geq 1).$$

- For $\mathfrak{f}$ α-admissible,

$$\mathfrak{f}(z) = \sum_{j=0}^{\infty} D^{\alpha} f(j) \Delta^{-\alpha} \mathcal{Z}(j), \quad z \in \mathbb{D},$$

converges absolutely and uniformly on compact subsets of $\mathbb{D}$.

α-Bounded approximate identities in $A^\alpha(\mathbb{D})$

- For $\mathfrak{f}$ α-admissible and $n \in \mathbb{N}$, set $\mathfrak{g}_n(z) = \dfrac{1}{\mathfrak{f}(z)} \sum_{j=0}^{n-1} D^\alpha f(j) \Delta^{-\alpha} \mathcal{Z}(j) \in A^\alpha(\mathbb{D})$

Theorem

(i) *If* $\mathfrak{f}(1) < \infty$ *then* $\lim_{n\to\infty} \mathfrak{g}_n(z) = 1$ *in* $A^\alpha(\mathbb{D})$.

(ii) *If* $\mathfrak{f}(1) = \infty$ *then* $\|\mathfrak{g}_n\|_{A^\alpha(\mathbb{D})} \leq 2$ *for every* n.

(iii) *If* $(1-z)\mathfrak{f}(z) \in A^\alpha(\mathbb{D})$ *and* $D^\alpha f(j) j^\alpha \to 0$ *as* $j \to \infty$, *then*

$$\lim_{n\to\infty} (1-z)\mathfrak{g}_n(z) = 1 - z \text{ in } A^\alpha(\mathbb{D}).$$

Proof.- Put $(1-z)\mathfrak{g}_n(z) = \mathfrak{h}_n(z) - \mathfrak{r}_n(z)$, where

$$\mathfrak{h}_n(z) = \frac{1}{\mathfrak{f}(z)} \left(D^\alpha f(0) + \sum_{j=1}^{n} D^\alpha f(j-1) k^\alpha(j) - \sum_{j=1}^{n-1} D^{\alpha+1} f(j-1) \Delta^{-\alpha} \mathcal{Z}(j) \right),$$

$$\mathfrak{r}_n(z) := \frac{1}{\mathfrak{f}(z)} D^\alpha f(n-1) \Delta^{-\alpha} \mathcal{Z}(n)$$

PROOF approximate identities

- $\lim_{n\to\infty} \|\mathfrak{h}_n - (1-z)\|_{A^\alpha(\mathbb{D})} = 0 \quad$ and $\quad \lim_{n\to\infty} \|\mathfrak{r}_n\|_{A^\alpha(\mathbb{D})} = 0.$

- For $\mathfrak{f}$, $\mathfrak{h}$ α-admissible such that $\mathfrak{h}, \mathfrak{f}\mathfrak{h} \in A^\alpha(\mathbb{D})$ and $m \in \mathbb{N}_0$,

$$\begin{aligned} W^\alpha(f*h)(m) &= \sum_{j=0}^{m} \sum_{l=m-j}^{m} k^\alpha(l+j-m) D^\alpha f(j) W^\alpha h(l) \\ &- \sum_{j=m+1}^{\infty} \sum_{l=m+1}^{\infty} k^\alpha(l+j-m) D^\alpha f(j) W^\alpha h(l) \end{aligned}$$

- Let $q \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Then

$$k^\alpha(q+m) = -\sum_{p=0}^{q-1} k^\alpha(p) \sum_{j=q-p}^{m+q-p} k^{-\alpha}(j) k^\alpha(m+q-p-j).$$

- Ex. 0-admissible sequences: classical Kaluza's theorem ([HaaseTomilov])
- Log-convex sequences of higher difference order: A sequence $f \neq 0$ is logarithmically convex of degree $m \in \mathbb{N}_0$ if for every $p \in \{0, 1, \ldots, m\}$ and $j \geq 0$ one has

$$D^p f(j) > 0 \ \text{ and } \ \left(D^p f(j+1)\right)^2 \leq D^p f(j) D^p f(j+2), \ \forall j \geq 0.$$

If, moreover, f is decreasing then $D^\beta f(j) > 0$ for $j \geq 0$ and $0 \leq \beta \leq m$.

- Kaluza's extension:

Theorem

Let $\mathfrak{f} = \sum_{j=0}^{\infty} f(j) z^j$ be a holomorphic function on the unit disc $\mathbb{D}$ such that $f := (f(j))_{j=0}^{\infty}$ is decreasing, log-convex of degree m, with $W^{-\beta}(D^\beta f) = f$ and $\lim_{j\to\infty} D^\beta f(j) j^\beta = 0$, $0 \leq \beta \leq m$. Then $\mathfrak{f}$ is zero-free on $\mathbb{D}$ and, for $g \longleftrightarrow \dfrac{1}{\mathfrak{f}}$,

$$W^\beta g(0) > 0 \quad \text{and} \quad W^\beta g(j) < 0, \quad j \geq 1, \ 0 \leq \beta \leq m.$$

That is, $\mathfrak{f}$ is β-admissible for each $0 \leq \beta \leq m$ (therefore $\frac{1}{\mathfrak{f}} \in A^m(\mathbb{D})$).

1) Let $0 < s < 1$. The function $\mathfrak{k}^s(z) = (1-z)^{-s}$ is α-admissible for all $\alpha \geq 0$:

– $1/\mathfrak{k}^s = \mathfrak{k}^{-s} \in A^\alpha(\mathbb{D})$ and k^s is bounded.

– $D^\alpha k^s(n) = \dfrac{\Gamma(1-s+\alpha)\Gamma(s+n)}{\Gamma(s)\Gamma(1-s)\Gamma(n+\alpha+1)} > 0$, $n \in \mathbb{N}_0$, so $\text{lím}_{n\to\infty} D^\alpha k^s(n) n^\alpha = 0$.

– $W^\alpha k^{-s}(n) = D^\alpha k^{-s}(n) = \dfrac{\Gamma(1+s+\alpha)\Gamma(-s+n)}{\Gamma(-s)\Gamma(1+s)\Gamma(n+\alpha+1)} < 0$, $n \in \mathbb{N}$,

– $W^\alpha k^{-s}(0) = \displaystyle\sum_{l=0}^{\infty} k^{-\alpha}(l) k^{-s}(l) = \dfrac{\Gamma(s+\alpha+1)}{\Gamma(1+s)\Gamma(1+\alpha)} > 0$.

– Also,

$$\begin{aligned} W^{-\alpha} D^\alpha k^s(n) &= \frac{1}{\Gamma(s)\Gamma(1-s)} \sum_{l=0}^{\infty} k^\alpha(l) \frac{\Gamma(1-s+\alpha)\Gamma(s+n+l)}{\Gamma(n+l+\alpha+1)} \\ &= \frac{1}{\Gamma(s)\Gamma(1-s)} \int_0^1 (1-x)^{s+n+1} x^{-s}\, ds = \frac{\Gamma(s+n)}{\Gamma(s) n!} = k^s(n). \end{aligned}$$

– Further, $(1-z)\mathfrak{k}^s(z) = \mathfrak{k}^{s-1}(z) \in A^\alpha(\mathbb{D})$ and $D^\alpha k^s(n) n^\alpha \to 0$ as $n \to \infty$.

Altogether $\mathfrak{k}^s$ satisfies Theorem on “admissible”BAI.

2) The function $\mathfrak{L}(z) = \dfrac{-\log(1-z)}{z} = \displaystyle\sum_{j=0}^{\infty} \frac{z^j}{j+1} = \sum_{j=0}^{\infty} L(j) z^j, \ z \in \mathbb{D}$,
is log-convex of any order m, and α-admissible for all $\alpha \geq 0$:

– $D^m L(j) = \dfrac{m!}{(j+1)\ldots(j+m+1)}$, so $(D^m L(j+1))^2 < D^m L(j) D^m f(j+2)$

– Also,

$$\begin{aligned} W^{-\alpha} D^{\alpha} L(j) &= \sum_{l=0}^{\infty} k^{\alpha}(l) \int_0^1 (1-x)^{j+l} x^{\alpha}\, ds \\ &= \int_0^1 (1-x)^j \, ds = f(j). \end{aligned}$$

– Further, $(1-z)\mathfrak{L}(z) = 1 - \displaystyle\sum_{j=1}^{\infty} \frac{z^j}{j(j+1)} \in A^{\alpha}(\mathbb{D})$ and $D^{\alpha} L(n) j^{\alpha} \to 0$ as $n \to \infty$.

Altogether $\mathfrak{L}$ satisfies Theorem on “admissible”BAI.

• Domain of operators $\mathfrak{f}(T)$, transferring (using FC) above results for (C,α)-bounded T and α-admissible $\mathfrak{f}$. Put $\mathfrak{g}_n(z) = \frac{1}{\mathfrak{f}(z)} \sum_{j=0}^{n-1} D^\alpha f(j) \Delta^{-\alpha} \mathcal{Z}(j) \in A^\alpha(\mathbb{D})$

Proposition

Assume $Ker(I-T) = \{0\}$.

(i) *If* $\mathfrak{f}(1) < \infty$ *then* $\lim_{n\to\infty} \mathfrak{g}_n(T) = I$ *in the operator norm.*

(ii) *If* $\mathfrak{f}(1) = \infty$ *then* $\sup_n \|\mathfrak{g}_n(T)\| < \infty$.

(iii) *If* $\mathfrak{f}(1) = \infty$, $(1-z)\mathfrak{f}(z) \in A^\alpha(\mathbb{D})$ *and* $D^\alpha f(j) j^\alpha \to 0$ *as* $j \to \infty$, *then*

$$\lim_{n\to\infty} (I-T)\mathfrak{g}_n(T) = I - T \quad \textit{in the operator norm.}$$

Proof. Straightforward from BAI theorem, $T = \Phi_\alpha(z \mapsto z)$, just making $z \longleftrightarrow T$.

• If $\overline{\mathrm{Ran}}(I-T) = X$ then $\mathfrak{g}_n(T)x \to x$ $(x \in X)$.
Also, in (iii), $\mathfrak{f}$ is α-regularizable by $1-z$ so that there exists $\mathfrak{f}(T)$ (as closed operator); in particular $\mathfrak{g}_n(T)\mathfrak{f}(T) \subset (\mathfrak{g}_n\mathfrak{f})(T)$.

Theorem

Let $\alpha > 0$ and let $\mathfrak{f}$ be an α-admissible function such that $(1-z)\mathfrak{f}(z) \in A^{\alpha}(\mathbb{D})$ and $D^{\alpha} f(j) j^{\alpha} \to 0$ as $j \to \infty$. If T is a (C,α)-bounded operator on X with $\overline{\mathrm{Ran}}(I-T) = X$, the following assertions are equivalent for a given x in X:

(i) $x \in \mathrm{Dom}\, \mathfrak{f}(T)$.

(ii) *The series* $\sum_{j \geq 0} D^{\alpha} f(j) \Delta^{-\alpha} \mathcal{T}(j) x$ *converges in norm.*

(iii) *The series* $\sum_{j \geq 0} D^{\alpha} f(j) \Delta^{-\alpha} \mathcal{T}(j) x$ *converges weakly.*

(iv) $\sup_N \| \sum_{j=0}^{N} D^{\alpha} f(j) \Delta^{-\alpha} \mathcal{T}(j) x \| < \infty$ *(only if X is reflexive).*

Furthermore, if (i)-(iii) holds true then

$$\mathfrak{f}(T)x = \sum_{j \geq 0} D^{\alpha} f(j) \Delta^{-\alpha} \mathcal{T}(j) x.$$

Proof. $\sum_{j=0}^{n-1} D^{\alpha} f(j) \Delta^{-\alpha} \mathcal{T}(j) x = (\mathfrak{g}_n \mathfrak{f})(T)x = \mathfrak{g}_n(T)\mathfrak{f}(T)x \to_{n \to \infty} \mathfrak{f}(T)x.$

- Recall, $(I-T)^s u = x$ where x is given and u is the unknown.

Theorem

Let T be (C,α)-bounded with $\overline{(I-T)X} = X$ and let $0 < s < 1$. For $x \in X$,

$$x \in \mathrm{Ran}(I-T)^s \Longleftrightarrow \sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha-s}} \Delta^{-\alpha}\mathcal{T}(n)x \text{ converges in norm (weakly) in } X.$$

In this case, $$(I-T)^{-s}x = \frac{\sin(\pi s)\Gamma(1-s+\alpha)}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{\Gamma(n+\alpha+1)} \Delta^{-\alpha}\mathcal{T}(n)x.$$

Corollary

Let T be (C,α)-bounded on $X = \overline{\mathrm{Ran}}(I-T)$. Then $\lim_{n\to\infty} M_T^{\alpha+1}(n)x = 0$ $(x \in X)$. Moreover, if $x \in \mathrm{Ran}(I-T)^s$ with $0 < s < 1$ then

$$\|M_T^{\alpha+1}(n)x\| = o(n^{-s}), \qquad \text{as } n \to \infty.$$

Theorem

Let T *be a* (C,α)*-bounded operator on* X *with* $\overline{\mathrm{Ran}}(I-T)=X$. *Given* $x\in X$,

$$x\in \mathrm{Dom}(\log(I-T)) \iff \sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}}\Delta^{-\alpha}\mathcal{T}(n)x \text{ converges in norm (or weakly).}$$

and then, with $c_\alpha=-\int_0^1(1-u^\alpha)(1-u)^{-1}du$,

$$-\log(I-T)x = H_T^{(\alpha)}x := c_\alpha x+\sum_{n=1}^{\infty}\frac{\Gamma(\alpha+1)\Gamma(n)}{\Gamma(n+\alpha+1)}\Delta^{-\alpha}\mathcal{T}(n)x$$

Proof. For $z\in\mathbb{D}$, $\log(1-z)=-\sum_{n\geq 1}n^{-1}z^j=\mathfrak{h}(z)-\mathfrak{L}(z)$ where

$$\mathfrak{h}(z):=1-\sum_{n=1}^{\infty}[n(n+1)]^{-1}z^n,\quad \mathfrak{L}(z):=\sum_{n=0}^{\infty}(n+1)^{-1}z^n;$$

with $\mathfrak{h}\in A^\alpha(\mathbb{D})$ and $\mathfrak{L}$ α-admissible. Hence, $\log(I-T)=\mathfrak{h}(T)-\mathfrak{L}(T)$.
Also,

$$\log(I-T)x=-T\mathfrak{L}(T)x=-\sum_{n=0}^{\infty}D^\alpha L(n)T\Delta^{-\alpha}\mathcal{T}(n)x=\ldots$$

CASE $0 < \alpha < 1$. BAI from TAYLOR COEFFICIENTES

Define $\mathfrak{g}_n^0(z) := (1-z)^s \sum_{j=0}^{n-1} k^s(j) z^j$, so $(1-z)\mathfrak{g}_n^0(z) = (1-z)\mathfrak{g}_n(z) - \mathfrak{s}_n(z)$ where

$$\mathfrak{s}_n(z) = \frac{(1-z)}{\mathfrak{f}(z)} \sum_{l=0}^{n-1} \Delta^{-\alpha}\mathcal{Z}(l) \sum_{j=n}^{\infty} k^{-\alpha}(j-l) f(j), \qquad |z| \le 1.$$

Theorem

Take $s \in (0,1)$ *and* $0 \le \alpha < 1 - s$. *Then* $\sup_n \|\mathfrak{g}_n^0\|_{A^\alpha(\mathbb{D})} < \infty$ *and* $\lim_{n\to\infty} (1-z)\mathfrak{g}_n^0(z) = 1 - z$ *in* $A^\alpha(\mathbb{D})$.

Proof. For the *sup*, careful study of signs and use of Γ. For the *limit*,

$$\begin{aligned} \mathfrak{s}_n(z) &= \frac{1}{\mathfrak{f}(z)}\left(\sum_{j=n}^{\infty} k^{-\alpha}(j) f(j) + \sum_{l=1}^{n} k^{\alpha}(l) \sum_{j=n}^{\infty} k^{-\alpha}(j-l+1) f(j) \right) \\ &- \frac{1}{\mathfrak{f}(z)} \sum_{l=1}^{n} \Delta^{-\alpha}\mathcal{Z}(l) \sum_{j=n}^{\infty} k^{-\alpha-1}(j-l+1) f(j) \\ &- \frac{1}{\mathfrak{f}(z)} \Delta^{-\alpha}\mathcal{Z}(n) \sum_{j=n}^{\infty} k^{-\alpha}(j-n+1) f(j) \end{aligned}$$

CASE $0 < \alpha < 1$. POISSON EQUATION

Theorem

Take $s \in (0,1)$, $0 \leq \alpha < 1 - s$. *Let* T *be* (C,α)*-bounded,* $\overline{(I-T)X} = X$. *Then*

$$x \in \mathrm{Ran}(I-T)^s \iff \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} T^n x \text{ converges in norm (or weakly)} .$$

In this case, $(I-T)^{-s}x = \sum_{n=0}^{\infty} k^s(n) T^n x$.

Corollary

Take $s \in (0,1)$, $0 \leq \alpha < 1 - s$. *Let* T *be* (C,α)*-bounded,* $\overline{(I-T)X} = X$. *Then*

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} T^j x = 0 \quad \forall x \in X.$$

Moreover,

$$\|\frac{1}{n}\sum_{j=1}^{n} T^j x\| = o(n^{-s}), \text{ as } n \to \infty, \forall x \in \mathrm{Ran}(I-T)^s.$$

CASE $0 < \alpha < 1$. Discrete HILBERT TRANSFORM

Recall, $\mathfrak{L}(z) := \sum_{n=0}^{\infty}(n+1)^{-1}z^n$, $|z| \le 1$. Put

$$\mathfrak{g}_n^L(z) := \frac{1}{\mathfrak{L}(z)} \sum_{j=0}^{n-1} \frac{1}{j+1} z^j, \quad |z| \le 1.$$

Theorem

Let α be such that $0 < \alpha < 1$. Let T be a (C,α)-bounded operator on X with $\overline{\mathrm{Ran}}(I-T) = X$. For a given $x \in X$ the following are equivalent:

(i) $x \in \mathrm{Dom}(\log(I-T))$.

(ii) *The series* $\sum_{n=1}^{\infty} \frac{1}{n} T^n x$ *converges (in norm or weakly).*

In any of the above cases

$$-\log(I-T)x = H_T x := \sum_{n=1}^{\infty} \frac{1}{n} T^n x.$$

VOLTERRA OPERATOR

- Let $1 \le p < \infty$. The Volterra operator is V, with

$$Vf(t) := \int_0^t f, \quad t \in [0,1],\ f \in L^p(0,1).$$

Define $T_V := I - V$. Power estimates of T_V (Hille 1945 for $p = 1$, Montes-Sánchez-Zemanek 2005); T_V power-bounded on $L^p(0,1) \Leftrightarrow p = 2$.

- Ergodic means (Hille), for $\alpha \ge 0$ ($\alpha = 0$ gives T_V^n),

$$M_{T_V}^{\alpha}(n)f = (\delta_0 - [k^{\alpha+1}(n)]^{-1} L_{n-1}^{(\alpha+1)}) \star f, \quad n \in \mathbb{N},\ f \in L^p(0,1),$$

where $L_{n-1}^{(\alpha+1)}$ = generalized Laguerre polynomial of degree $n-1$.

Hille $\Rightarrow$ T_V is (C,α)-bounded on $L^1(0,1)$ if and only if T_V is (C,α)-ergodic on $L^1(0,1)$ if and only if $\alpha > 1/2$.

- Partial extension for every $1 < p < \infty$.

$$\Lambda_{n-1}^{(\alpha+1)} := [k^{\alpha+1}(n)]^{-1} L_{n-1}^{(\alpha+1)}, \quad n \in \mathbb{N}$$

Proposition

(i) *T_V is (C,α)-bounded and (C,α)-ergodic on $L^p(0,1)$ $\forall \alpha > |1/2 - 1/p|$.*

(ii) *Take $s \in (0,1)$ in the Volterra integral equation (V-eq)*

$$\frac{1}{\Gamma(s)} \int_0^t (t-u)^{s-1} g(u)\, du = f(t), \quad 0 \le t \le 1, f \in L^p(0,1).$$

- *Solution:* $g = M_s \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!} (\delta_0 - \Lambda_{n-1}^{(\alpha+1)}) \star f \in L^p(0,1)$

  and $\|(\delta_0 - \Lambda_{n-1}^{(\alpha+2)}) \star f\|_p = o(n^{-s})$, *as* $n \to \infty$.

- *If, moreover,* $0 < s < 1 - |1/2 - 1/p|$ *and* $\alpha \in (|1/2 - 1/p|, 1-s)$,

$$g = \sum_{n=0}^{\infty} k^s(n)(\delta_0 - L_{n-1}^{(1)}) \star f \in L^p(0,1).$$

and $\frac{1}{n} \| \sum_{j=1}^{n} (\delta_0 - \Lambda_{j-1}^{(1)}) \star f\|_p = o(n^{-s})$, *as* $n \to \infty$.

Proposition

(iii) *Let* $\alpha > |1/2 - 1/p|$. *Then*

- $f \in \mathrm{Dom}(\log V) \subset L^p(0,1)$ *if and only if* $\sum_{n=1}^{\infty} n^{-1}(\delta_0 - \Lambda_{n-1}^{(\alpha+1)}) * f$ *is convergent. In this case,*

$$(\log V)f = (\psi(\alpha+1) - \psi(1))f - \sum_{n=1}^{\infty} \frac{1}{n}(\delta_0 - \Lambda_{n-1}^{(\alpha+1)}) \star f.$$

- *If moreover* $\alpha \in (|1/2 - 1/p|, 1)$,

$$f \in \mathrm{Dom}(\log V) \subset L^p(0,1) \Longleftrightarrow \sum_{n=1}^{\infty} n^{-1}(\delta_0 - L_{n-1}^{(1)}) \star f \text{ is convergent.}$$

In this case,

$$(\log V)f = -\sum_{n=1}^{\infty} \frac{1}{n}(\delta_0 - L_{n-1}^{(1)}) \star f.$$

SHIFT OPERATOR

Let $0<\beta<1$. Let T_S be the backward shift on $\ell^2_\beta(\mathbb{N}_0)$, that is,

$$(T_S f)(j) = f(j+1), \quad j \in \mathbb{N}_0$$

for every f such that $\|f\|^2_{2,\beta} := \sum_{j=0}^{\infty} |f(j)|^2 k^\beta(j) < \infty$.

- One has $\|T_S^n\|^2 \sim (n+1)^{1-\beta}$, but also that
  T_S is (C,α)-bounded for $\alpha > (1-\beta)/2$ (Abadías-Bello-Yakubovich),
  whence T_S is (C,α)-ergodic for $\alpha > (1-\beta)/2$.
- $I - T_S$ is the first order finite difference operator $W = D$, that is,

$$(I-T_S)f(n) = f(n) - f(n+1), \qquad n \in \mathbb{N}_0,$$

(Poisson eq. in discrete difference equations: $((I-T)^s)_{s>0}$ useful tool in diff. eq. of fractional order, discretization techniques in problems on differential equations)
D^2 = discretization of Laplacian; $D^{2s}u = f$ (Markov chains theory).

- Recently,
  - Maximum and comparison principles, uniqueness, as well as a probabilistic interpretation for $D^s u = f$ (Abadías-De León-Torrea).
- Elements of $(I-T)^s X$ = proper data in problems on difference equations (inverse problem).

Proposition

Assume $\alpha > (1-\beta)/2$, $0 < s < 1$.

(i) *The elliptic problem in differences* $D^s u = f \in \ell^2_\beta(\mathbb{N}_0)$ *has solution*

$$u = M_s \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{\Gamma(n+\alpha+1)} \sum_{j=0}^{n} k^\alpha(n-j) f(j+\cdot).$$

If, moreover, $0 < s < (1+\beta)/2$ *and* $\alpha \in ((1-\beta)/2, 1-s)$ *then*

$$u = \sum_{n=0}^{\infty} k^s(n) f(n+\cdot) = W^{-s} f.$$

(ii) *For* $f \in \mathrm{Dom}(\log D) \subset \ell^2_{k^\beta}(\mathbb{N}_0)$ *if and only if the series*

$$((\log D)f =) c_\alpha - \sum_{n=1}^{\infty} B(\alpha+1, n) \sum_{j=0}^{n} k^\alpha(n-j) f(j+\cdot) \text{ converges.}$$

If, moreover, $\alpha \in ((1-\beta)/2, 1)$ *then*

$$(\log D) f = -\sum_{n=1}^{\infty} \frac{1}{n} f(n+\cdot).$$

THANK YOU FOR YOUR ATTENTION