An overview of exceptional orthogonal polynomials

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Classical orthogonal polynomials

If $y_0, y_1, y_2, \ldots$ are polynomials with $\deg y_n = n$ that satisfy

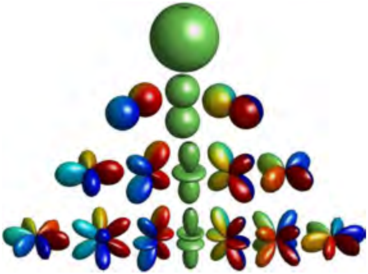
$$p(x)y_i'' + q(x)y_i' + r(x)y_i = \lambda_i y_i, \qquad i = 0, 1, 2, \ldots$$

and $y_0, y_1, y_2, \ldots$ form an orthogonal polynomial system (OPS), then

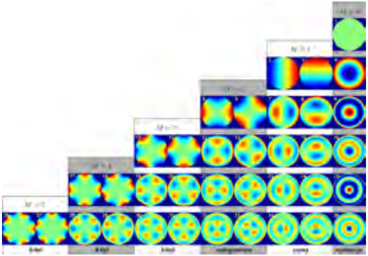
1. p, q, r polynomials and $\deg p \leq 2$, $\deg q \leq 1$ and $\deg r = 0$.
2. $\{y_i\}_{i=0}^{\infty}$ are Hermite, Laguerre or Jacobi polynomials.

Classical polynomials: ubiquitous applications

- Basis of a Hilbert space composed of very simple functions
- Suitable for numerical analysis, interpolation, approximating functions, etc.
- They appear in the solution of many interesting problems in physics



Spherical harmonics



Zernike polynomials

(special cases of Jacobi polynomials)

Bochner's paper (1929)

Über Sturm-Liouvillesche Polynomsysteme.

Von

S. Bochner in München.

Wir betrachten irgendeine Differentialgleichung der Form

$$(1) \qquad p_0(x)\,y'' + p_1(x)\,y' + p_2(x)\,y + \lambda\,y = 0\,.$$

Die Koeffizienten $p_0(x)$, $p_1(x)$, $p_2(x)$ sind irgendwelche reell- oder komplexwertige Funktionen der Variablen x; von denen wir in erster Linie nur anzunehmen brauchen, daß sie in einem gemeinsamen Intervall J der x-Achse definiert sind; und λ bedeutet einen Parameter, der aller komplexen Werte fähig ist.

Wir setzen nunmehr voraus, daß es eine Folge von Parameterwerten

$$(2) \qquad \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$$

und eine Folge von Polynomen

$$(3) \qquad P_0(x), P_1(x), P_2(x), \ldots, P_n(x), \ldots$$

gibt, von denen das n-te, $P_n(x)$, von *genau* n-tem Grade ist, und die so beschaffen sind, daß das Polynom $P_n(x)$ im Intervall J der Gleichung (1) für den Wert $\lambda = \lambda_n$ genügt:

Bochner's paper (1929)

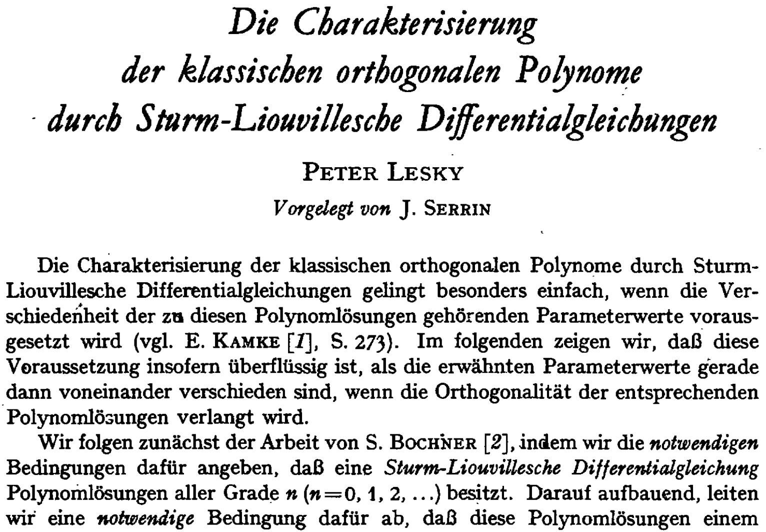
$$(4) \qquad p_0(x)\,P_n''(x) + p_1(x)\,P_n'(x) + p_2(x)\,P_n(x) + \lambda_n\,P_n(x) = 0.$$

Ein solches System von Polynomen wollen wir ein Sturm-Liouvillesches Polynomsystem nennen. Von solcher Art sind die wichtigsten Polynomsysteme der mathematischen Physik: die Legendreschen (allgemeiner: die hypergeometrischen), die Laguerreschen und die Hermiteschen Polynome (vgl. etwa: Courant-Hilbert, Methoden der mathematischen Physik I, Kap II, § 8 und § 10).

Es ist eine sehr leichte Aufgabe, die Gesamtheit *aller* Sturm-Liouvilleschen Polynomsysteme aufzustellen. Da aber diese Aufstellung in der

"It is a very easy task to describe the set of *all* Sturm-Liouville polynomial systems..."

Lesky's paper (1962)


Die Charakterisierung der klassischen orthogonalen Polynome durch Sturm-Liouvillesche Differentialgleichungen

PETER LESKY

Vorgelegt von J. SERRIN

Die Charakterisierung der klassischen orthogonalen Polynome durch Sturm-Liouvillesche Differentialgleichungen gelingt besonders einfach, wenn die Verschiedenheit der zu diesen Polynomlösungen gehörenden Parameterwerte vorausgesetzt wird (vgl. E. KAMKE [*1*], S. 273). Im folgenden zeigen wir, daß diese Voraussetzung insofern überflüssig ist, als die erwähnten Parameterwerte gerade dann voneinander verschieden sind, wenn die Orthogonalität der entsprechenden Polynomlösungen verlangt wird.

Wir folgen zunächst der Arbeit von S. BOCHNER [*2*], indem wir die *notwendigen* Bedingungen dafür angeben, daß eine *Sturm-Liouvillesche Differentialgleichung* Polynomlösungen aller Grade n ($n=0, 1, 2, \ldots$) besitzt. Darauf aufbauend, leiten wir eine *notwendige* Bedingung dafür ab, daß diese Polynomlösungen einem

Lesky's paper (1962)

8. Zum Abschluß wollen wir kurz auf die Behauptung eingehen, daß die klassischen orthogonalen Polynome die *einzigen* orthogonalen Polynomlösungen Sturm-Liouvillescher Differentialgleichungen sind.

Der erste Beweis dieser Behauptung ist eigentlich bereits in einer Arbeit von W. HAHN [8] enthalten. HAHN sucht in der genannten Arbeit die orthogonalen Polynome aller Grade n ($n=0, 1, 2, \ldots$) — also Polynomketten —, die folgende Eigenschaft haben: Die ersten Ableitungen dieser Polynome bilden im gleichen Intervall bezüglich einer geänderten Gewichtsfunktion ebenfalls ein Orthogonalsystem. Für derartige Polynomketten leitet er eine Differential-

To conclude we briefly address the claim that the classical orthogonal polynomials are *the only* orthogonal polynomial solutions of a differential equation of Sturm-Liouville type.

Are they REALLY all ?

Exceptional orthogonal polynomials

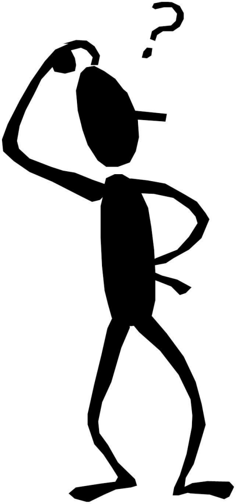
If $y_0, y_1, y_2, \ldots$ are polynomials with $\deg y_n = n$ that satisfy

$$p(x)y_i'' + q(x)y_i' + r(x)y_i = \lambda_i y_i, \qquad i \in \sigma = \mathbb{N} - \{i_1, \ldots, i_m\}$$

and $\{y_i\}_{i\in\sigma}$ form an orthogonal polynomial system (OPS), then

1. p, q, r could be rational functions
2. There are new, **complete** systems of orthogonal polynomials $\longrightarrow$ exceptional orthogonal polynomials (X-OPS)

Notation:
σ is the degree sequence= a numerable subset of $\mathbb{N}$
m is the codimension = the number of gaps (missing integers) in the degree sequence.



Two immediate questions

1. If y_i are polynomials and $q(x)$ and $r(x)$ are rational functions in

$$\underbrace{p(x)y_i'' + q(x)y_i' + r(x)y_i}_{rational} = \underbrace{\lambda_i y_i}_{polynomial},$$

$\rightarrow$ some cancellations must happen for each i
infinite values of i but p, q, r are fixed. . . try to find an example !

2. If there are a finite number of missing degrees. . . can the remaining polynomials span the basis of a Hilbert space ?

Plan of the talk

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   - Rational solutions to Painlevé equations
   - Monodromy free potentials
   - Quantum superintegrable systems
   - Schur polynomials and Jacobi-Trudi formulas

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6. **Open Problems**

The X-men

- Milson, Grandati, Kamran
- Sasaki, Odake, Ho
- Quesne, Marquette
- Durán, Pérez, Varona
- Marcellán, Kuijlaars, Kasman
- Takemura
- Miki, Tusjimoto, Vinet
- Post, Winternitz
- Littlejohn, Liaw, Stewart
- Hallnäs, Veselov,
- Zhedanov
- Horvath
- Schulze-Halberg, Roy
- Yadav, Khare, Mandal
- Dimitrov, Lun
- Pupasov-Maximov

Definition

A countable sequence of polynomials $\{y_i \,:\, i \in \sigma\}$ forms an **exceptional orthogonal polynomial system** if the following conditions are met:

1. there exists a 2nd order diff. oper. $T = pD_{zz} + qD_z + r$ such that
$$T[y_i] = \lambda_i y_i, \qquad i \in \sigma = \mathbb{N} - \{i_1, \dots, i_m\}$$
2. (*regularity*) there exists a positive definite weight W such that
$$\int_{\mathbb{R}} y_i\, y_j W\, dz = C_{ij}\delta_{i,j}, \qquad \forall i, j \in \sigma$$
3. (*density*) The set $\{y_i\}_{i\in\sigma}$ spans a basis of $\mathrm{L}_2(\mathbb{R}, W)$.

The simplest X-Hermite example

Define the following polynomials:

$$y_n = \mathrm{Wr}(H_1(x), H_2(x), H_n(x)), \qquad n \in \sigma, \quad \sigma = \{0, 3, 4, 5, \dots\}$$

They satisfy the following eigenvalue equation

$$-y_n'' + 2\left(x + \frac{4x}{1+2x^2}\right) y_n' = 2n y_n, \qquad n \in \sigma$$

The fist few of them are:

$$\begin{aligned}
y_0(x) &= \mathrm{Wr}(H_1(x), H_2(x), H_0(x)) = 16 \\
y_3(x) &= \mathrm{Wr}(H_1(x), H_2(x), H_3(x)) = 128x^3 + 192x \\
y_4(x) &= \mathrm{Wr}(H_1(x), H_2(x), H_4(x)) = 768x^4 + 762x^2 - 192 \\
y_5(x) &= \mathrm{Wr}(H_1(x), H_2(x), H_5(x)) = 3072x^5 - 3840x
\end{aligned}$$

The simplest X-Hermite example

Weight:

$$W(x) = \frac{e^{-x^2}}{(4+8x^2)^2}, \qquad x \in \mathbb{R}.$$

Orthogonality relation

$$\langle y_n, y_m \rangle \equiv \int_{\mathbb{R}} y_n y_m \, W \, dx = \sqrt{\pi}\, 2^{n+2}\, n!(n-1)(n-2)\, \delta_{n,m}$$

Completeness

The set $\{y_n\}_{n\in\sigma}$ spans a basis of $L_2(\mathbb{R}, W)$.

Differential constraint

$$\text{span}\{y_n\}_{n\in\sigma} = \left\{ p \in \mathbb{R}[x] \quad \text{s.t.} \quad 2xp' - (2x^2+1)p \Big|_{x=\pm\frac{i}{\sqrt{2}}} = 0 \right\}$$

Towards a full classification

Can we give a full classification *à la Bochner* of exceptional polynomials ?

Conjecture (2012)

Every exceptional polynomial system can be obtained by applying a finite number of Darboux transformations to a classical polynomial system

If conjecture holds, classification becomes a constructive process:

1. Start from classical polynomials
2. Apply the most general sequence of rational Darboux transformations
3. Select the admissible ones (those leading to a regular weight)

DGU, Kamran, Milson, FoCM 2012.

Darboux transformations

- Start from $H = -D_{xx} + u$
- Pick a formal eigenfunction $\phi \to$ we can factorize H as:

$$H = BA \quad \text{where} \begin{cases} A = D_x - \frac{\phi'}{\phi} \\ B = -D_x - \frac{\phi'}{\phi} \end{cases}$$

- *Darboux transformed* operator $\hat{H}$

$$\hat{H} = AB = -D_{xx} + \hat{u}, \qquad \hat{u} = u - 2(\log \phi)_{xx}$$

- Intertwining relations: $AH = \hat{H}A \quad HB = B\hat{H}$
- We can construct eigenfunctions of $\hat{H}$ by applying A:

$$\left.\begin{array}{l} H\psi = \lambda\psi \\ \hat{\psi} = A\psi \end{array}\right\} \Rightarrow \hat{H}\hat{\psi} = \lambda\hat{\psi}$$

Iterated Darboux transformations

- Apply Darboux transform iteratively

$$H_0 = H \xrightarrow{\phi_1} H_1 = \hat{H} \xrightarrow{A(\phi_2)} H_2 = \hat{\hat{H}} \longrightarrow \cdots$$

$$\begin{aligned} H_j &= A_j B_j + \lambda_j, \quad j = 1, \ldots, n-1 \\ &= B_{j+1} A_{j+1} + \lambda_{j+1} \end{aligned}$$

- Higher order intertwining relations:

$$H_0 \mathcal{B} = \mathcal{B} H_n, \qquad \text{where} \quad \mathcal{B} = B_1 B_2 \cdots B_n$$
$$\mathcal{A} H_0 = H_n \mathcal{A}, \qquad \text{where} \quad \mathcal{A} = A_n \cdots A_2 A_1.$$

- Wronskian representation (Crum Formula)

$$\mathcal{A}(y) \propto \frac{\mathrm{Wr}(\phi_1, \ldots, \phi_n, y)}{\mathrm{Wr}(\phi_1, \ldots, \phi_n)}$$

Rational Darboux transformations

- Spectral properties of the transformation (Deift '79)
  1. *State-deleting*: $\phi \in L^2$ (ground state of H)
  2. *State-adding*: $\phi^{-1} \in L^2$
  3. *Isospectral*: Neither ϕ nor ϕ^{-1} are L^2

- **Rational Darboux transformations** are the subset of DTs that map polynomials into polynomials,

$$\psi_n = \mu(x) \cdot P_n(z(x)) \;\longrightarrow\; \hat{\psi}_n = \hat{\mu}(x) \cdot \hat{P}_n(z(x)), \quad P_n, \hat{P}_n \text{ polynomials}$$

- A Darboux transf. is **rational** iff the seed function ϕ has the form

$$\phi(x) = \mathrm{e}^{\sigma(z(x))} f(z(x)), \quad (\log f)' \text{ is a rational function of } z$$

Rational DTs for Hermite polynomials

- Harmonic oscillator
$$V(x;\omega) = \frac{\omega^2}{4}x^2 - \frac{\omega}{2}$$
- Bound states (State-deleting)
$$\phi_n = \psi_n(x,\omega) = \mathrm{e}^{-z^2} H_n(z), \quad z = \sqrt{\frac{\omega}{2}}x, \quad E_n = \omega n$$
- Negative energy states (State-adding)
The reflection $\omega \to -\omega$ is a discrete symmetry (up to spectral shift):
$V(x;-\omega) = V(x;\omega) + \omega$.
$$\phi_{-(n+1)} \equiv \psi_n(x,-\omega) = \mathrm{e}^{z^2} \mathrm{i}^n H_n(\mathrm{i}z), \quad z = \sqrt{\frac{\omega}{2}}x, \quad E_{-(n+1)} = -n\omega$$

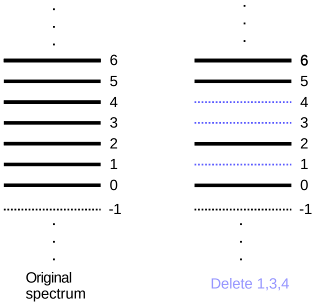
Rational DTs for Hermite polynomials

Delete states at energies 1,3,4:

$$V_0 = \frac{\omega^2}{4}x^2 - \frac{\omega}{2}$$

$$\phi_n = \mathrm{e}^{-z^2} H_n(z)$$

$$V_{1,3,4} = V_0 - 2D_{xx} \log \mathrm{Wr}(\phi_1, \phi_3, \phi_4)$$



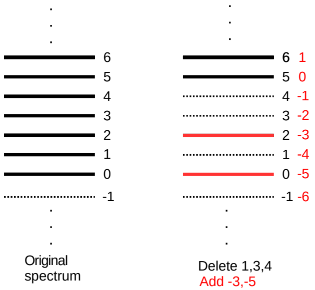
Equivalent Darboux transformations

Add states at energies -3 and -5:

$$V_0 = \frac{\omega^2}{4}x^2 - \frac{\omega}{2}$$

$$\phi_{-(n+1)} = \mathrm{e}^{z^2}\mathrm{i}^n H_n(\mathrm{i}z)$$

$$V_{-3,-5} = V_0 - 2D_{xx}\log \mathrm{Wr}(\phi_{-3},\phi_{-5})$$



Odake 2014, DGU, Grandati, Milson 2014

Maya diagrams and partitions

$$V_{1,3,4}(x) = V_{-3,-5}(x) - 5\omega$$

Both potentials coincide up to spectral shift $\Rightarrow$ eigenfunctions coincide
Pulling out e^{-x^2} factors we have an identity of the form

$$\mathrm{Wr}[H_1, H_3, H_4] = \text{const} \times \mathrm{Wr}[H_2(\mathrm{i}x), H_4(\mathrm{i}x)]$$

Maya diagrams

$$\begin{array}{cc} \ldots\text{oo}|\text{xoxooxx}\ldots & \ldots\text{ooxoxoo}|\text{xx}\ \ldots \\ (1,3,4) & (-3,-5) \end{array}$$

But we can also mix adding and deleting transformations:

$$\begin{array}{c} \ldots\text{ooxo}|\text{xooxx}\ldots \\ (-2,1,2) \end{array}$$

Takemura 2014

Pseudo-Wronskians

$$\begin{aligned} V_{1,3,4}(x) &= V_{-3,-5}(x) - 5\omega \\ &= V_{-2,1,2} - 3\omega \end{aligned}$$

...ooooo|xoxooxxxxxx...	...oooooxoxoo|xxxxxx ...	...ooooooxo|xooxxxxxx...
$(1,3,4)$	$(-3,-5)$	$(-2,1,2)$

$$\begin{aligned} \mathrm{Wr}[H_1, H_3, H_4] &= \text{const} \times \mathrm{Wr}[H_2(\mathrm{i}x), H_4(\mathrm{i}x)] \\ &= \text{const} \times \tilde{\mathrm{Wr}}[H_1(\mathrm{i}x) \mid H_1(x), H_2(x)] \end{aligned}$$

$\tilde{\mathrm{Wr}}$ is a *pseudo-Wronskian*, defined as follows:

Pseudo-Wronskians

$$\tilde{\mathrm{Wr}}[H_i(\mathrm{i}x) \,|\, H_j(x), H_k(x)] = \det \begin{pmatrix} H_i(\mathrm{i}x) & H_j(x) & H_k(x) \\ U H_i(\mathrm{i}x) & H_j'(x) & H_k'(x) \\ U^2 H_i(\mathrm{i}x) & H_j''(x) & H_k''(x) \end{pmatrix}$$

where

$$\begin{aligned} U[H_n(\mathrm{i}z)] &= -\mathrm{i}H_{n+1}(\mathrm{i}z), \\ H_n'(z) &= 2nH_{n-1}(z) \end{aligned}$$

Example

$$\det \begin{pmatrix} H_1(x) & H_3(x) & H_4(x) \\ H_1'(x) & H_3'(x) & H_4'(x) \\ H_1''(x) & H_3''(x) & H_4''(x) \end{pmatrix} = \det \begin{pmatrix} H_1(\mathrm{i}x) & H_1(x) & H_2(x) \\ -\mathrm{i}H_2'(\mathrm{i}x) & H_1'(x) & H_2'(x) \\ (\mathrm{i})^2 H_3''(\mathrm{i}x) & H_1''(x) & H_2''(x) \end{pmatrix}$$

A quick guide to X-Hermite polynomials

- Pick any sequence of Hermite polynomials: $H_{k_1}, \dots, H_{k_\ell}$
- **First observation**: the Wronskian determinant satisfies a linear second order differential equation (in fact, ℓ different equations).

$$H_\lambda := \mathrm{Wr}[H_{k_1}, \dots, H_{k_\ell}]$$

- **Second observation**: Now freeze all Hermite polynomials but one, and let the last one range over all other indices:

$$H_{\lambda,j} := \mathrm{Wr}[H_{k_1}, \dots, H_{k_\ell}, H_j] \quad j \notin \{k_1, \dots, k_\ell\}$$

- For a given $\lambda = (k_1, \dots, k_\ell)$, you obtain a numerable sequence of polynomials where some degrees are missing. Let's call them **exceptional Hermite polynomials** associated to the partition λ.

Three main properties

1. X_λ-Hermites are **eigenfunctions** of a differential equation

$$T_\lambda[y] := y'' - 2\left(x + \frac{H'_\lambda}{H_\lambda}\right) y' + \left(\frac{H''_\lambda}{H_\lambda} + 2x\frac{H'_\lambda}{H_\lambda}\right) y$$

$$T_\lambda\left[H_{\lambda,j}\right] = 2(\ell - j)\, H_{\lambda,j}, \qquad j \notin \{k_1, \dots, k_\ell\}$$

   Spectrum has "missing" eigenvalues at levels $k_1, \dots, k_\ell$.

2. **Orthogonality**

$$\int_{-\infty}^{\infty} H_{\lambda,i} H_{\lambda,j}\, W_\lambda(x)\, dx = \delta_{i,j} \sqrt{\pi}\; 2^j\, j!\, p_\lambda(j),$$

$$W_\lambda(x) = \frac{e^{-x^2}}{\left(H_\lambda(x)\right)^2}, \qquad p_\lambda(x) = (x - k_1)(x - k_2)\cdots(x - k_\ell)$$

3. **Completeness**: the set $\left\{H_{\lambda,j} \,|\, j \in \mathbb{N}\backslash\{k_1, \dots, k_\ell\}\right\}$ is a basis of $L^2(\mathbb{R}, W_\lambda)$.

An example

First few X_λ-Hermite polynomials for $\lambda = (1,1,3,3) \longrightarrow k = (1,2,5,6)$

$$H_{\lambda,0} = \mathrm{Wr}[H_1, H_2, H_5, H_6, H_0] \propto 4x^4 + 3$$
$$H_{\lambda,3} = \mathrm{Wr}[H_1, H_2, H_5, H_6, H_3] \propto 8x^7 + 36x^5 + 30x^3 + 15x$$
$$H_{\lambda,4} = \mathrm{Wr}[H_1, H_2, H_5, H_6, H_4] \propto 16x^8 + 80x^6 + 120x^4 + 60x^2 - 15$$
$$H_{\lambda,7} = \mathrm{Wr}[H_1, H_2, H_5, H_6, H_7] \propto 32x^{11} + 112x^9 + 192x^7 + 336x^5 - 210x^3 - 315x$$

$$W_\lambda = \frac{e^{-x^2}}{\mathrm{Wr}[H_1, H_2, H_5, H_6]^2} = \frac{e^{-x^2}}{(15 + 60x^2 + 16x^6 + 16x^8)^2}$$

Weights

$\lambda = (1,1,3,3) \longrightarrow k = (1,2,5,6)$

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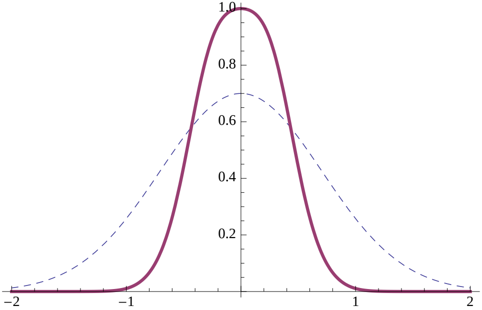
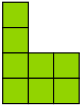


$$W_\lambda = \frac{e^{-x^2}}{\mathrm{Wr}[H_1,H_2,H_5,H_6]^2} \propto \frac{e^{-x^2}}{(15+60x^2+16x^6+16x^8)^2}$$

Potentials



$$k = (1, 2, 5, 6)$$

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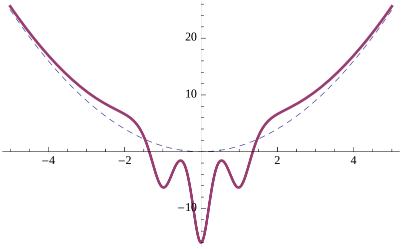
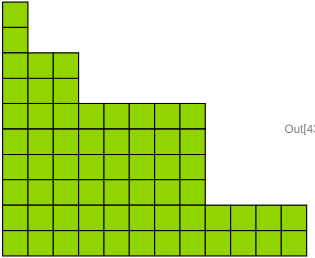


$$V_\lambda(x) = x^2 - 2D_{xx} \log \mathrm{Wr}[H_1, H_2, H_5, H_6]$$

$$V_\lambda(x) = x^2 + \frac{16\left(-225 + 900x^2 - 900x^4 - 2640x^6 - 3120x^8 + 192x^{10} + 320x^{12} + 256x^{14}\right)}{(15 + 60x^2 + 16x^6 + 16x^8)^2}$$

Wiggly Potentials

$$k = (1, 2, 5, 6, 12, 13, 14, 15, 20, 21)$$



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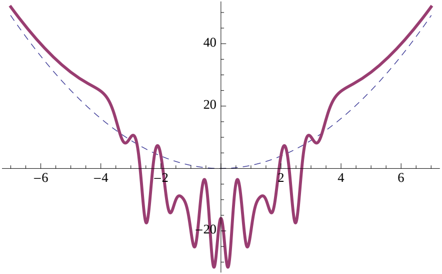

$$V_\lambda(x) = x^2 - 2D_{xx} \log \mathrm{Wr}[H_1, H_2, H_5, H_6, H_{12}, H_{13}, H_{14}, H_{15}, H_{20}, H_{21}]$$

Eigenfunctions

$$\lambda = (1,1,3,3) \longrightarrow k = (1,2,5,6)$$

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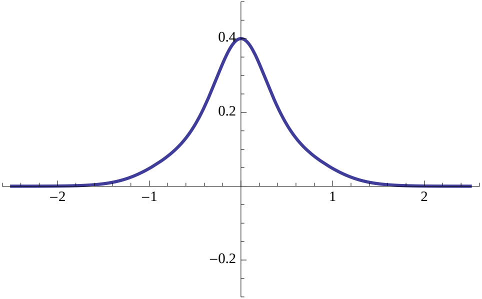
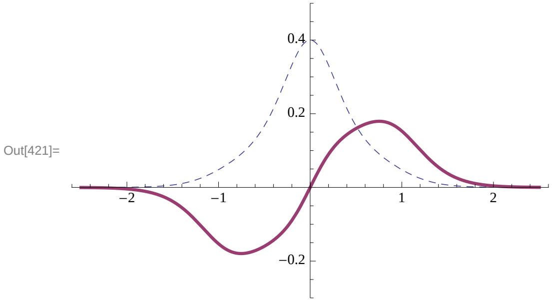


$$\phi_0 = \frac{e^{-x^2}}{\mathrm{Wr}[H_1, H_2, H_5, H_6]} (4x^4 + 3)$$

Eigenfunctions

$$\lambda = (1,1,3,3) \longrightarrow k = (1,2,5,6)$$

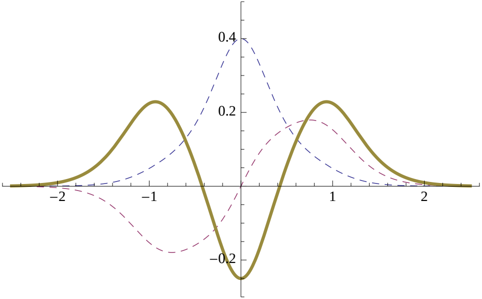


$$\phi_1 = \frac{e^{-x^2}}{\mathrm{Wr}[H_1, H_2, H_5, H_6]} (8x^7 + 36x^5 + 30x^3 + 15x)$$

Eigenfunctions

$\lambda = (1,1,3,3) \longrightarrow k = (1,2,5,6)$

Out[422]=

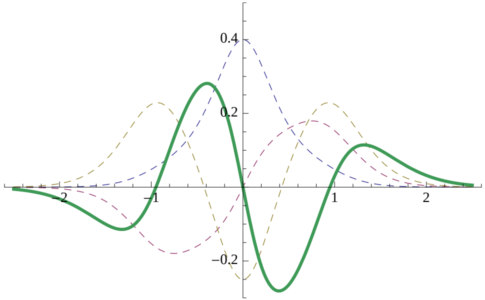


$$\phi_2 = \frac{e^{-x^2}}{\mathrm{Wr}[H_1, H_2, H_5, H_6]} (16x^8 + 80x^6 + 120x^4 + 60x^2 - 15)$$

Eigenfunctions

$$\lambda = (1, 1, 3, 3) \longrightarrow k = (1, 2, 5, 6)$$

Out[423]=



$$\phi_3 = \frac{e^{-x^2}}{\mathrm{Wr}[H_1, H_2, H_5, H_6]} (32x^{11} + 112x^9 + 192x^7 + 336x^5 - 210x^3 - 315x)$$

Admissible sequences

- A sequence or partition $(k_1, \ldots, k_\ell)$ is *admissible*
  $\Leftrightarrow \quad H_\lambda(x) = \mathrm{Wr}[H_{k_1}, \ldots, H_{k_\ell}] \neq 0$ for all $x \in \mathbb{R}$.
- Only admissible sequences lead to well defined weights .
- Admissible sequences for X-Hermite:

$$\{1,2,3,4\} \cup \{7,8\} \qquad \checkmark$$

$$\{1,2,3\} \cup \{5,6,7,8\} \qquad \times$$

$$\{0,1,2\} \cup \{5,6,7,8\} \cup \{10,11\} \qquad \checkmark$$

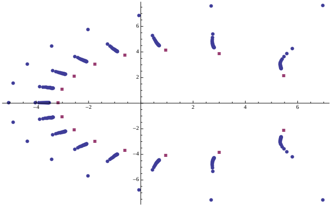
Krein 1957, Adler 1992, Durán 2014

Regular and exceptional zeros

A polynomial of degree 11 has only 3 real zeros, where are the extra zeros ?

Exceptional orthogonal polynomials have two types of zeros:

1. regular zeros lie in the support of the measure, they interlace and follow the same asymptotics as the zeros of classical polynomials.
2. exceptional zeros lie outside the support of the measure, for fixed codimension m and large degree n they converge to the zeros of the Wronskian denominator.



Exceptional zeros of X-Laguerre $L^{(\alpha)}_{m,m+j}$ for $m = 15$ and $j = 1, \ldots, 22$.

DGU,Marcellán, Milson 2012

Kuijlaars & Milson 2015

What have we learned so far ?

Classical orthogonal polynomials

What have we learned so far ?

Classical orthogonal polynomials

Exceptional orthogonal polynomials

Recursion relations

Classical polynomials have 3-term recurrence relations (c.f. Jacobi matrix)
Exceptional polynomials have two types of recurrence relations:

Type I

- order $2\ell + 3$
  ℓ is $\#$ Darboux steps.
- Coeffs. are polys in n and x.
- different from the usual formulas
- explicit formulas for recurrence coefficients

Type II

- order $2m + 3$
  m is $\#$ of gaps (codimension)
- Coeffs. are rational fn's of n.
- similar to Jacobi matrix
- bispectral algorithm to compute recurrence coefficients

Recursion relations - Type I

The polynomials $H_{\lambda,j}$ satisfy the following $2\ell+3$-term recursion relation

$$\sum_{k=0}^{2\ell+2} B^{\ell}_{n,k} H_{\lambda,n+k}, \quad n \geq -\ell-1,$$

$$B^{\ell}_{n,k} = \sum_{j=\lceil \frac{k}{2} \rceil}^{\min(k,\ell+1)} (-1)^k 2^{-j} (n+k+1)_{\ell+1-j} C^{\ell+1}_{2j-k,k-j} H_{2j-k}$$

C^n_{ij}	trinomial coefficient
$(x)_i$	Pochhammer symbol
H_k	k^{th} Hermite polynomial
ℓ	length of λ = number of Darboux steps

Odake 2013, DGU,Grandati, Milson 2014

Type I Recursion relations

Take $\ell = 2$, $\lambda = (1,2)$, the X-Hermite polynomials $H_{\lambda,n}$ satisfy a 7-order recursion relation

$$\begin{aligned} &8(n+1)(n+2)(n+3)\, H_{\lambda,n} - 24(n+2)(n+3)x\, H_{\lambda,n+1} \\ &\quad +12(n+3)(2x^2+n+3)\, H_{\lambda,n+2} - \big(8x^3 + 12x(2n+7)\big)\, H_{\lambda,n+3} \\ &\quad +6(2x^2+n+4)\, H_{\lambda,n+4} - 6x\, H_{\lambda,n+5} + H_{\lambda,n+6} = 0. \end{aligned}$$

The initial values for the sequence are:

$$H_{\lambda,0} = 16, \qquad H_{\lambda,1} = 0, \qquad H_{\lambda,2} = 0.$$

Type II Recursion relations

Define $f_\lambda(x) = \int^x H_\lambda, \quad d = \deg f_\lambda$

$$f_\lambda(x) H_{\lambda,n}(x) = \sum_{j=-d}^{d} A_{\lambda,j}(n) H_{\lambda,n+j}, \qquad n \geq 0$$

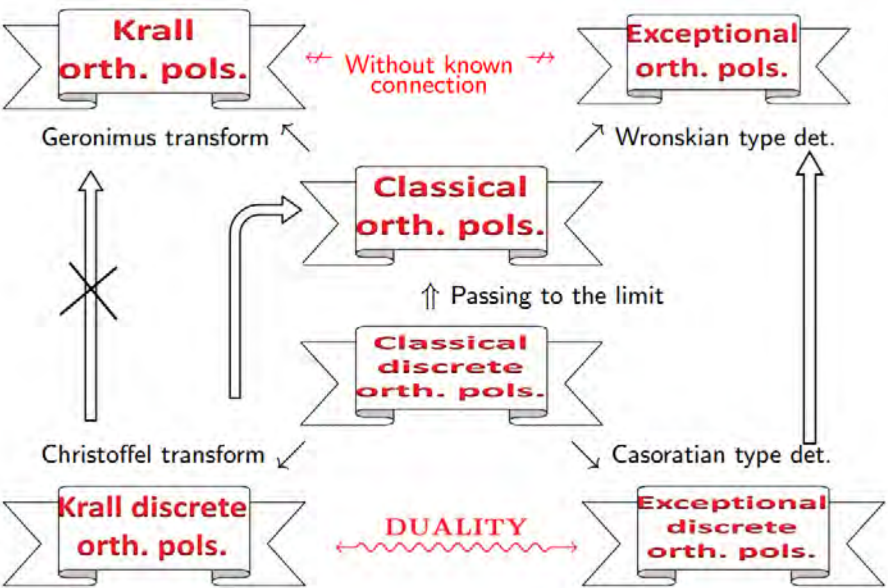
where $A_{\lambda,j}(n)$ is a rational function of n, does not depend on x.
For instance, if $\lambda = (1,1) \leftrightarrow k = (1,2)$, we have

$$A_{\lambda,j} = \begin{cases} 4n(n-1)(n-2)/3 & \text{if } j = -3 \\ 2n(n-1) & \text{if } j = -1 \\ n-2 & \text{if } j = 1 \\ \frac{(n-1)(n-2)}{6(n+1)(n+2)} & \text{if } j = 3 \\ 0 & \text{if } j = -2, 0, 2. \end{cases}$$

Jacobi matrix is 7-diagonal

Durán, Miki & Tsujimoto 2014, Odake 2015

An extended Askey scheme



Rational solutions to Painlevé equations

- N-Periodic Darboux chains $\rightarrow$ rational solutions of Painlevé equations
  $N=3$: solutions to PIV (Okamoto pols., generalized Hermite pols.).
  $N=4$: solutions to PV (Umemura pols.)
  $N\geq 3$: solutions to higher order Painlevé equations of type A_{N-1}
- Generalized Hermite polynomials

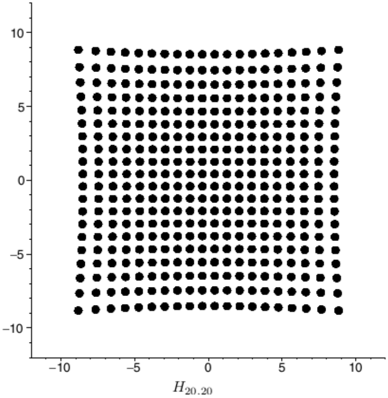
  Sequence $\{m+j\}_{j=0}^{n-1} \quad \rightarrow \quad H_{m,n}=\mathrm{Wr}(H_m,H_{m+1},\dots,H_{m+n})$

  $(\log H_{m,n})'$ is a rational solution to PIV
- Okamoto polynomials
  Sequence $\{1+3k\}_{k=0}^{M}\cup\{2+3j\}_{j=0}^{N}$

Veselov & Shabat 1993, Adler 1994, Noumi& Yamada 1999

Regular patterns of zeros

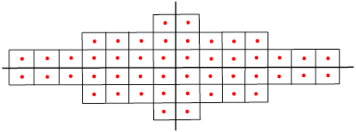
Complex zeros of $H_{20,20} = \mathrm{Wr}[H_{20}, H_{21}, \ldots, H_{39}]$

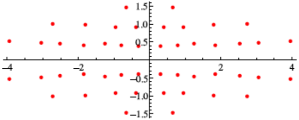


Clarkson 2006

Zeros of Wronskians of Hermite polynomials

Complex zeros of $H_\lambda(x)$ for $\lambda = (1,1,4,4,7,7)$





Conjecture

All the zeros (real or complex) of

$$H_\lambda(x) = \mathrm{Wr}[H_{k_1}, \ldots, H_{k_n}]$$

are **simple**, except for $x = 0$.

Felder, Hemery, Veselov 2012

Main open problem

Besides being nice...

Are exceptional orthogonal polynomials useful ?

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i.e. What kind of functions, arising in what kind of problems admit a better approximation in a basis of XOPs than in a different basis ?

Idea: All basis elements in an XOP basis satisfy a given set of differential constraints.
If the function we want to approximate must satisfy such constraints (for some reason) , then it is likely that an expansion in terms of XOPs will be more efficient (since it encodes the constraints in a natural manner).

Other open problems

- Combinatorial aspects of X-polynomials
- Recursion relations $\rightarrow$ Jacobi matrices $\rightarrow$ Spectral theory
- New ensembles in Random Matrix Theory
- Extension to multivariate case.

Gracias por su atención