Holomorphic Semigroups and Fractional Calculus.

Valentin Keyantuo

University of Puerto Rico, Río Piedras Campus

Zaragoza

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Plan

- Introduction
- Hadamard-type fractional integrals
- Holomorphy of semigroups generated by the fractional Laplacian

We consider some analytic semigroups appearing in fractional calculus. The semigroup related to the Hadamard fractional integral in relation to the time derivative, and the semigroup generated by the fractional Laplace operator in the case of the space derivative.
From **joint work with Ahmed Sani, Omar El-Mennaoui (Agadir), Fabian Seoanes, Mahamadi Warma (San Juan)**.

The most common fractional integral is the Riemann-Liouville fractional integral: for $\alpha > 0$,

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ t > 0.$$

Replacing α with $z \in \mathbb{C}$, $Re(z) > 0$ the family $(I^z)_{Re(z)>0}$ is a holomprphic semigroup acting on the space $L^p[0, 1]$, $1 \leq p < \infty$.

This semigroup is related to the the tranlation semigroup on the interval [0, 1]. When $\alpha = 1$, we have the Volterra operator. The above fractional power semigroup was studied extensively in the monograph by Hille and Phillips (1957).
One important fact is the the semigroup $(I^z)_{Re(z)>0}$ has a boundary value group on the imaginary axis. The case $p = 2$ is to be found in Hille-Phillips (1957). For general $1 < p < \infty$, this was done by M. J. Fischer (*Proc. AMS*, 1967) and G. K. Kalish (*Amer. J. Math.*, 1971).

More recently, Arendt-El-Mennaoui and Hieber (*Proc. AMS*, 1997) obtained this result independently. They obtain the result on the boundary values by using the Coifman-Weiss transference principle. We adopt the latter for the consideration of the Hadamard fractional integral.

In the articles Butzer-Kilbas-Trujillo (*J. Math. Anal, Appl.*, 2002a, 2002b) and Kilbas (2001), the Hadamard fractional integral was considered along with a generalization and the semigroup property was established. The Hadamard fractional integral first appeared in the paper "*Essai sur les fonctions données par leur développement de Taylor*" by J. Hadamard (*J. Math Pures Appl.*, 1892).

The (generalized) Hadamard fractional integral of a function f is defined as follows:

$$(\mathcal{J}_\mu^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (\frac{t}{x})^\mu (\ln(\frac{x}{t}))^{\alpha-1} f(t) \frac{dt}{t} \quad (x > 0), \qquad (2.1)$$

where $\alpha > 0$ and $\mu \in \mathbb{R}$. The original definition corresponds to $\mu = 0$ and is discussed in the monograph by Samko-KIlbas-Marichev (1993, [Chapter 4, Section 18.3]).

As in the papers by Butzer-Kilbas-Trujillo cited above, we consider the Banach spaces

$$X_c^p = \{f : (0,1) \longrightarrow \mathbb{C}, f \quad \text{measurable} \quad \text{and} \quad \|f\|_{c,p} = \left(\int_0^1 |x^c f(x)|^p \frac{dx}{x}\right)^{1/p} < \infty\}.$$

The space X_c^p is a Banach space and if $c = \frac{1}{p}$, it coincides with $L^p(0,1)$. We note that if $\alpha = \mu = 1$ then $(\mathcal{J}_1^1 f)(x) = \frac{1}{x}\int_0^x f(t)dt$, $x > 0$, which is the Cesàro operator, so that L^p boundedness of $\mathcal{J}_1^1$ is to be compared to Hardy's inequality. It follows that the semigroup $(\mathcal{J}_1^\mu)$ represents the fractional powers of the Cesàro operator.

Theorem

Let $1 \leq p < \infty$ and $\mu, c \in \mathbb{R}$ with $\mu > c$. Then the family of operators $(\mathcal{J}_\mu^\alpha)_{\alpha>0}$ acting on the space X_c^p forms a strongly continuous semigroup which has an analytic extension to the right half-plane $\mathbb{C}_+ = \{\alpha \in \mathbb{C} \mid \mathrm{Re}(\alpha) > 0\}$. Moreover, the semigroup $(\mathcal{J}_\mu^\alpha)_{\alpha>0}$ has a boundary C_0-group. More precisely, $(\mathcal{J}_\mu^{\mathrm{is}})_{\mathrm{s}\in\mathbb{R}}$, given by

$$(\mathcal{J}_\mu^{\mathrm{is}}\mathrm{f}) = \lim_{\sigma\to 0^+} (\mathcal{J}_\mu^{\sigma+\mathrm{is}}\mathrm{f}), \ \forall \mathrm{f} \in \mathrm{X}_\mathrm{c}^\mathrm{p} \tag{2.2}$$

and $(\mathcal{J}_\mu^{\mathrm{is}})_{\mu\in\mathbb{R}}$ forms a C_0-group, provided $1 < \mathrm{p} < \infty$.

Proof

We consider the operator family

$$(T(t)f)(x) = e^{-\mu t} f(e^{-t}x),\ x \in (0, 1),\ t > 0. \tag{2.3}$$

Then it is readily verified that $T = (T(t))_{t \geq 0}$ is a strongly continuous group on the space X_c^p defined above. The infinitesimal generator of T is the operator $A = -x\dfrac{d}{dx} - \mu I$. If $\mu > c$ then T is exponentially stable. In fact,

$$\begin{aligned} \|T(t)f\|^p_{X_p^c} &= \int_0^1 e^{-\mu p t} |x^c f(e^{-t}x)|^p \frac{dx}{x} \\ &= e^{-p(\mu - c)t} \int_0^{e^{-t}} |u^c f(u)|^p \frac{du}{u} \\ &\leq e^{-p(\mu - c)t} \|f\|^p_{X_p^c}. \end{aligned}$$

Proof

The fractional powers $A^{-\alpha}$ for $\alpha > 0$ are given by the well-known formula:

$$((-A)^{-\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}(T(t)f)(x)dt, \; f \in X. \qquad (2.4)$$

By a change of variable in the integral, we have for $f \in X$:

$$\begin{aligned} ((-A)^{-\alpha}f)(x) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}(T(t)f)(x)dt \\ &= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}e^{-\mu t}f(e^{-t}x)dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^x (\frac{t}{x})^\mu (\ln(\frac{x}{t}))^{\alpha-1} f(t)\frac{dt}{t} \end{aligned}$$

Proof

which is (2.3). From this representation, the semigroup property for the family $(\mathcal{J}_{\mu}^{\alpha})_{\alpha>0}$ follows by the general theory of fractional powers of operators. Analyticity also follows from the general theory.

In order to obtain the last assertion, we note that $(\mathrm{T}(\mathrm{t}))$ is a strongly continuous semigroup of positive contraction operators on the space $\mathrm{X} = \mathrm{L}^{\mathrm{p}}(0, 1)$, $1 < \mathrm{p} < \infty$. The conclusion is obtained by application of the Coifman-Weiss transference principle (see Coifman-Weiss (1976), or the monograph by Arendt-Batty-Hieber-Neubrander (2011)).

Let us consider the particular case where $\mu = 1$.

Corollary

Assume that $1 < p < \infty$. *Then the the family*

$$T(z)f(x) = ((-A)^{-z}f)(x) = \frac{1}{x\Gamma(z)} \int_0^x (\ln(\frac{x}{t}))^{z-1} f(t)dt,$$

$x > 0$, $\mathrm{Re}(z) > 0$ *forms a holomorphic semigroup of angle* $\frac{\pi}{2}$ *in the space* $L^p(0,1)$. *This semigroup admits a boundary value group on the imaginary axis.*

Another consequence of the above representation is the explicit description of the powers of the averaging operator of the *Cesàro operator* C,

$$(\mathrm{Cf})(\mathrm{x}) := \frac{1}{\mathrm{x}} \int_0^{\mathrm{x}} \mathrm{f}(\mathrm{s})\mathrm{ds},\ \mathrm{f} \in \mathrm{L}^{\mathrm{p}}(0, 1).$$

Clearly $\mathrm{C} = \mathrm{T}(1)$. The strong continuity of C in $\mathrm{L}^{\mathrm{p}}(0, 1)$, $1 < \mathrm{p} < \infty$ yields the Hardy's inequality (without the constant).

Corollary

For each $n \in \mathbb{N}$ *and* $f \in L^p(0, 1)$ *we have*

$$(C^n f)(x) = \frac{1}{x(n-1)!} \int_0^x (\ln(\frac{x}{t}))^{n-1} f(t) dt \quad . \qquad (2.5)$$

This is of course a direct consequence of the semigroup property: $C^n = (T(1))^n = T(n)$.

We observe that this formula was obtained by as Lemma 2 in D. W. Boyd (*Pacific. J. Math.*, 1968) who used mathematical induction. He used this result to study the spectral radius of averaging operators. The spectral theory of the Cesàro operator (including the discrete version) has been studied in several papers, (see for example Arendt-de Pagter (*Pacific. J. Math.*, 2002) where the Boyd indices are used in the description of the spectrum in various Banach function spaces).

Boyd obtains the following formula (we consider the case $a = 0$ in his formula)

$$(C^n f)(x) = \frac{1}{(n-1)!}\int_0^1 (\ln(\frac{1}{s}))^{n-1} f(sx)ds,\ f \in L^p(0, 1) \qquad (2.6)$$

which is readily obtained from (2.5) by a change of variable. In fact, the semigroup $(T(t))$ in Corollary 2.2 can be written as follows

$$(T(z)f)(x) = \frac{1}{\Gamma(z)}\int_0^1 (\sigma)^{\mu-1}(\ln(\frac{1}{\sigma}))^{z-1} f(\sigma x)d\sigma,\ f \in L^p(0, 1),$$

$\mathrm{Re}(z) > 0$. We observe that the above theorem and its corollaries remain valid if we replace $L^p(0, 1)$ with $L^p(0, a)$ where $a \in (0, \infty]$.

We state this below for the case $a = \infty$. For that we introduce the space

$$X_c^p := \{f : (0, \infty) \longrightarrow \mathbb{C}, f \text{ measurable and } \|f\|_{c,p} = \left(\int_0^\infty |x^c f(x)|^p \frac{dx}{x}\right)^{1/p} < \infty\}.$$

Theorem

Let $1 \leq p < \infty$, μ, $c \in \mathbb{R}$, *with* $\mu > c \in \mathbb{R}$ *Then the family of operators* $(\mathcal{J}_\mu^\alpha)_{\alpha>0}$ *acting on the space* X_c^p *forms a strongly continuous semigroup which has an analytic extension to the right half-plane* $\mathbb{C}_+$. *Moreover, the semigroup* $(\mathcal{J}_\mu^\alpha)_{\alpha>0}$ *has a boundary* $\mathcal{C}_0-$*group on* X_c^p *denoted* $(\mathcal{J}_\mu^{is})_{s\in\mathbb{R}}$ *where*

$$(\mathcal{J}_\mu^{is} f)(x) = \lim_{\sigma\to 0^+} (\mathcal{J}_\mu^{\sigma+is} f)(x) \tag{2.7}$$

provided $1 < p < \infty$.

For the remainder of this section, we consider a second form of the Hadamard fractional integral operator to which the above construction applies (see again Butzer-Kilbas-Trujillo and Samko-KIlbas-Marichev and for the case $\mu = 0$). Here we set

$$(\mathcal{I}_\mu^\alpha \mathrm{f})(\mathrm{x}) = \frac{1}{\Gamma(\alpha)} \int_{\mathrm{x}}^{\infty} (\frac{\mathrm{x}}{\mathrm{u}})^{\mu} (\ln(\frac{\mathrm{u}}{\mathrm{x}}))^{\alpha-1} \mathrm{f}(\mathrm{u}) \frac{\mathrm{du}}{\mathrm{u}}, \ \mathrm{x} > 0. \tag{2.8}$$

We obtain the following counterpart of Theorem 2.1.

Theorem

Let $1 \le \mathrm{p} < \infty$ and let $\mu \in \mathbb{R}$ such that $\mathrm{c} + \mu > 0$. Then the family of operators $(\mathcal{I}_\mu^\alpha)_{\alpha>0}$ acting on $\mathrm{X}_\mathrm{c}^\mathrm{p}$ forms a strongly continuous semigroup which has an analytic extension to the right half-plane $\mathbb{C}_+$. Moreover, the semigroup has a boundary C_0-group on on $\mathrm{X}_\mathrm{c}^\mathrm{p}$.

Proof

The proof proceeds similarly as in the previous case.

In the paper "The Hadamard Fractional Power in Mikhlin-Sobolev inclusions" (*Rev. Union Math. Argentina*, 2004), J. E. Galé carried out a study of Hadamard fractional powers with the perspective of functional calculus.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be an arbitrary open set and let $T = (T(t))_{t\geq 0}$ be a self-adjoint semigroup on $L^2(\Omega)$ with generator A. Assume that A is dissipative.
Assume that the semigroup T interpolates on $L^p(\Omega)$, $1 \leq p < \infty$. That is, there exists for each p, a strongly continuous semigroup T_p on $L^p(\Omega)$ with $T_2 = T$ such that $T_p(t)f = T_2(t)f$ for all $f \in L^p(\Omega) \cap L^2(\Omega)$. Using the Stein interpolation theorem, it has been shown for example in (E. B. Davies: *"Heat Kernels and Spectral Theory"* (1989)) that for $1 < p < \infty$, the semigroup T_p is holomorphic on $L^p(\Omega)$ of angle $\theta_p \geq \frac{\pi}{2}\left(1 - \left|\frac{2}{p} - 1\right|\right)$.

The case $p = 1$ is more delicate and has been solved by Ouhabaz in (*Proc. AMS*, 1995) in a general context. More precisely in this paper, the author has shown that if T has a Gaussian estimate for $0 \leq t \leq 1$ in the sense that there are two positive constants M and b such that

$$|T(t)f| \leq MG(bt)|f| \quad \text{for} \quad 0 \leq t \leq 1 \quad \text{and for all} \quad f \in L^2(\Omega), \quad (3.1)$$

where $G = (G(t))_{t \geq 0}$ is the Gaussian semigroup on $L^2(\mathbb{R}^N)$ (the semigroup generated by the Laplace operator Δ on $L^2(\mathbb{R}^N)$), then there exists $\omega \geq 0$ such that the semigroup $(e^{-\omega t}T_p(t))_{t \geq 0}$ is bounded holomorphic of angle $\frac{\pi}{2}$ on $L^p(\Omega)$ for $1 \leq p < \infty$.

Of concern in the present work is the investigation of the holomorphy of the associated semigroups in L^{p}. Again, the case of Gaussian estimates was studied in E. M. Ouhabaz (*Proc. AMS*, 1995). In our case, we consider fractional order operators.

We now describe the problem in more details.
For $0 < s < 1$ we let

$$\mathcal{L}^1_s(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \ \text{ measurable, } \int_{\mathbb{R}^N} \frac{|u(x)|}{(1+|x|)^{N+2s}}\, dx < \infty \right\}.$$

For $u \in \mathcal{L}^1_s(\mathbb{R}^N)$ and $\varepsilon > 0$ we set

$$(-\Delta)^s_\varepsilon u(x) := C_{N,s} \int_{\{y\in\mathbb{R}^N:\ |x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N,$$

where $C_{N,s}$ is a normalization constant given by

$$C_{N,s} := \frac{s 2^{2s} \Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}. \tag{3.2}$$

The fractional Laplacian $-(-\Delta)^s$ is defined through the following singular integral:

$$(-\Delta)^s u(x) := C_{N,s} \, \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy = \lim_{\varepsilon \downarrow 0} (-\Delta)^s_\varepsilon u(x), \quad x \in \mathbb{R}^N,$$

provided that the limit exists. We notice that $\mathcal{L}^1_s(\mathbb{R}^N)$ is the right space for which $v_\varepsilon := (-\Delta)^s_\varepsilon u$ exists for every $\varepsilon > 0$, v_ε being also continuous at the continuity points of u.

Let $(-\Delta)^s_{\mathbb{R}^N}$ be the operator on $L^2(\mathbb{R}^N)$ given by

$$D((-\Delta)^s_{\mathbb{R}^N}) := \Big\{u \in W^{s,2}(\mathbb{R}^N) : \ (-\Delta)^s u \in L^2(\mathbb{R}^N)\Big\},$$
$$(-\Delta)^s_{\mathbb{R}^N} u = (-\Delta)^s u.$$

Then $(-\Delta)^s_{\mathbb{R}^N}$ is the self-adjoint operator on $L^2(\mathbb{R}^N)$ associated with the symmetric, closed and bilinear form $\mathcal{F}$ with $D(\mathcal{F}) = W^{s,2}(\mathbb{R}^N)$ and given by

$$\mathcal{F}(u,v) = \frac{C_{N,s}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy. \quad (3.3)$$

We mention that $(-\Delta)^s_{\mathbb{R}^N})$ can also be defined as follows:

$$(-\Delta)^s_{\mathbb{R}^N} u = \frac{1}{\Gamma(-s)}\int_0^\infty \Big(G(t)u - u\Big)\frac{dt}{t^{1+s}}.$$

It is well-known that the operator $-(-\Delta)^s_{\mathbb{R}^N}$ generates a submarkovian (positivity-preserving and L^∞-contractive) semigroup $(e^{-t(-\Delta)^s_{\mathbb{R}^N}})_{t\geq 0}$ on $L^2(\mathbb{R}^N)$ which is ultracontractive in the sense that it maps $L^1(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$. More precisely, there is a constant $C > 0$ such that

$$\|e^{-t(-\Delta)^s_{\mathbb{R}^N}}\|_{\mathcal{L}(L^1(\mathbb{R}^N),L^\infty(\mathbb{R}^N))} \leq Ct^{-\frac{N}{2s}}, \ \ \forall\, t > 0. \tag{3.4}$$

In addition, the semigroup has a kernel $0 \leq P_s(t,\cdot,\cdot) \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying

$$\left(e^{-t(-\Delta)^s_{\mathbb{R}^N}} f\right)(x) = \int_{\mathbb{R}^N} P_s(t,x,y) f(y)\, dy,$$

for every $f \in L^2(\mathbb{R}^N)$. It has been shown in Blumenthal-Getoor (*TAMS*, 1960), Z-Q Chen-T. Kumagai (*Probab. Theory Related Fields*, (2008)) that the kernel P_s satisfies the following estimates: There are two constants $0 < C_1 \leq C_2$ such that for all $x, y \in \mathbb{R}^N$ and $t > 0$, we have

$$C_1 t^{-\frac{N}{2s}} \left(1 + |x-y| t^{-\frac{1}{2s}}\right)^{-(N+2s)} \leq P_s(t,x,y) \leq C_2 t^{-\frac{N}{2s}} \left(1 + |x-y| t^{-\frac{1}{2s}}\right)^{-(N+2s)}. \quad (3.5)$$

We notice that the semigroup $(e^{-t(-\Delta)^s_{\mathbb{R}^N}})_{t\geq 0}$ does not have a Gaussian estimate, that is, it does not satisfy a Gaussian estimate. Indeed, it is well-known that the kernel K_G of the Gaussian semigroup G is given by

$$K_G(t,x,y) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4t}} \text{ for all } x,y\in\mathbb{R}^N \text{ and } t>0. \tag{3.6}$$

Therefore if $(e^{-t(-\Delta)^s_{\mathbb{R}^N}})_{t\geq 0}$ (with $0<s<1$) had a Gaussian estimate, then there would exist two constants $b>0$ and $C>0$ such that

$$P_s(t,x,y) \leq \frac{C}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4bt}} \text{ for all } x,y\in\mathbb{R}^N \text{ and } t>0. \tag{3.7}$$

From the above two-sided estimate of $P_s(t,x,y)$, it is clear that a Gaussian estimate cannot be true.

We note that the semigroups $(e^{-t(-\Delta)^s_{\mathbb{R}^N}})_{t\geq 0}$ and $(G(t))_{t\geq 0}$ are related by a subordination principle. More precisely, for every $u \in L^2(\mathbb{R}^N)$, we have

$$e^{-t(-\Delta)^s_{\mathbb{R}^N}} u = \int_0^\infty f_{t,s}(\tau) G(\tau) u \, d\tau \quad t > 0, \tag{3.8}$$

where the function $f_{t,s}$ (known as the stable Lévy process) is the inverse Laplace transform of the function $e^{-t\lambda^s}$. For more details we refer to the book "Functional Analysis" by K. Yosida, and the nice paper ("*A subordination principle on Wright functions and regularized resolvent families*", *J. Funct. Spaces*, (2015)) by Abadias and Miana. In particular the latter reference gives the relationship between $f_{t,s}$ and the well-known Wright functions. It follows from (3.8) that the kernel P_s can be obtained from K_G as follows:

$$P_s(t,x,y) = \int_0^\infty f_{t,s}(\tau) K_G(\tau,x,y) \, d\tau, \quad x,y \in \mathbb{R}^N. \tag{3.9}$$

The fractional Laplace operator $-(-\Delta)^s_{\mathbb{R}^N}$ is known in the literature as the generator of the so-called s-stable Lévy process but the name of Lévy has not been given to the semigroup $(e^{-t(-\Delta)^s_{\mathbb{R}^N}})_{t\geq 0}$. Since $\lim_{s\uparrow 1^-}(-\Delta)^s = -\Delta$ in the sense that

$$\lim_{s\uparrow 1^-}\int_{\mathbb{R}^N} v(-\Delta)^s u\, dx = \int_{\mathbb{R}^N} \nabla u\cdot\nabla v\, dx = -\int_{\mathbb{R}^N} v\Delta u\, dx, \tag{3.10}$$

for every $u\in W^{2,2}(\mathbb{R}^N)$ and $v\in W^{1,2}(\mathbb{R}^N)$, we shall call $(e^{-t(-\Delta)^s_{\mathbb{R}^N}})_{t\geq 0}$ the **fractional Gaussian semigroup**. We mention that the proof of (3.10) is where the constant $C_{N,s}$ given in (3.2) plays a crucial role.

In analogy with (3.1) we introduce the following notion.

Definition

We shall say that a self-adjoint semigroup T on $L^2(\Omega)$ has a **fractional Gaussian estimate** for $0 \le t \le 1$ if there are two positive constants M and b, and some $s \in (0, 1)$ such that

$$|T(t)f| \le Me^{-bt(-\Delta)^s_{\mathbb{R}^N}}|f| \quad \text{for} \quad 0 \le t \le 1 \quad \text{and for all} \quad f \in L^2(\Omega). \tag{3.11}$$

If (3.11) holds for all $t \ge 0$, then we shall simply say that T has a fractional Gaussian estimate.

Remark

If T *is a submarkovian semigroup on* $L^2(\Omega)$*, then by [Theorem 1.4.1] (E. B. Davies, 1989), there exists a consistent family of semigroups* $T_p(t)$ *on* $L^p(\Omega)$ *such that* $T_p(t)f = T_2(t) := T(t)f$ *for* $f \in L^p(\Omega) \cap L^2(\Omega)$, $1 \le p \le \infty$. *In addition the semigroup* T_p *is strongly continuous in* $L^p(\Omega)$ *if* $1 \le p < \infty$.

Remark

Now assume that T *is a semigroup satisfying* (3.11). *Since the semigroup* $(e^{-t(-\Delta)^s_{\mathbb{R}^N}})_{t\geq 0}$ *is submarkovian, and hence, contractive on* $L^p(\mathbb{R}^N)$ *for* $1 \leq p \leq \infty$, *we have that if* (3.11) *holds, then there exist constants* $\omega \geq 0$ *and* $M \geq 1$ *such that the semigroup* T *satisfies* $\|T(t)f\|_{L^p(\Omega)} \leq Me^{\omega t}\|f\|_{L^p(\Omega)}$ *for* $f \in L^p(\Omega) \cap L^2(\Omega)$, $1 \leq p \leq \infty$. *By the Riesz-Thorin interpolation theorem, there exists* $T_p(t) \in \mathcal{L}(L^p(\Omega))$ *such that* $T_p(t)f = T_2(t)f := T(t)f$ *for* $f \in L^p(\Omega) \cap L^2(\Omega)$, Throughout the paper, for $0 < \psi < \pi$, we shall denote by $\Sigma(\psi)$ the sector

$$\Sigma(\psi) := \{\mathrm{z} = \mathrm{re}^{\mathrm{i}\alpha},\ \mathrm{r} > 0,\ \ |\alpha| < \psi\}$$

of the complex plane $\mathbb{C}$.

The following result is taken from the internet seminar notes by W. Arendt (2006, internet Seminar Notes, [Proposition 14.2.4]).

Proposition

Let $\Omega \subset \mathbb{R}^N$ *be an arbitrary open set. Let* $D \subset \mathbb{C}$ *be open and let the mapping* $\mathbb{S} : D \to \mathcal{L}(L^2(\Omega))$ *be holomorphic such that*

$$\sup_{z \in \mathbb{K}} \|\mathbb{S}(z)\|_{\mathcal{L}(L^1(\Omega), L^\infty(\Omega))} < \infty$$

for every compact set $\mathbb{K} \subset D$.

Proposition (continued)

Then there exists a function $\mathbb{F} : \mathrm{D} \times \Omega \times \Omega \to \mathbb{C}$ *satisfying*

$$\mathbb{F}(z, \cdot, \cdot) \in L^\infty(\Omega \times \Omega) \quad \textit{for all } z \in D,$$

$$(\mathbb{S}(z)f)(x) = \int_\Omega \mathbb{F}(z, x, y) f(y) dy \quad \textit{for a.e. } x \in \Omega \quad \textit{and}$$

$$\forall f \in L^1(\Omega) \cap L^2(\Omega),$$

$$\mathbb{F}(\cdot, x, y) : D \to \mathbb{C} \textit{ is holomorphic for all } x, y \in \Omega.$$

We recall the definition of spaces of homogeneous type. If $(\mathrm{E}, \mu, \mathrm{d})$ is a metric space (with distance d) endowed with a measure μ, then for $\mathrm{x} \in \mathrm{E}$, $\mathrm{r} > 0$, we denote by $\mathrm{B}(\mathrm{x}, \mathrm{r})$ the open ball in E with radius r and center at x. We say that $(\mathrm{E}, \mu, \mathrm{d})$ has the volume doubling property if there exists a constant $\mathrm{C} \geq 1$ such that for all $\mathrm{x} \in \mathrm{E}$, $\mathrm{r} > 0$, $|\mathrm{B}(\mathrm{x}, 2\mathrm{r})| \leq \mathrm{C}|\mathrm{B}(\mathrm{x}, \mathrm{r})|$ where we use the notation $\mu(\mathrm{B}(\mathrm{x}, \mathrm{r})) = |\mathrm{B}(\mathrm{x}, 2\mathrm{r})$. With this property, $(\mathrm{E}, \mu, \mathrm{d})$ is called a space of homogeneous type.

The following extension result is contained in (Duong-Robinson, *JFA* (1996)) as [Proposition 3.3].

Proposition

Let $(\mathrm{E}, \mu, \mathrm{d})$ *be a space of homogeneous type. Let* $0 < \psi \leq \frac{\pi}{2}$ *and let* $\mathrm{z} \in \Sigma(\psi) \mapsto \mathbb{K}(\mathrm{z}, \mathrm{x}, \mathrm{y}) \in \mathbb{C}$ $(\mathrm{x}, \mathrm{y} \in \mathrm{E})$ *be the kernel of a holomorphic family of bounded operators on* $\mathrm{L}^2(\mathrm{X}, \mu)$*. Assume that* $\mathbb{K}$ *satisfies the following estimates for some* $\mathrm{m} > 0$*:*

Proposition (continued)

1. *There is a constant* $\mathrm{C}_1 > 0$ *such that*

$$|\mathbb{K}(\mathrm{z}, \mathrm{x}, \mathrm{y})| \leq \mathrm{C}_1 |\mathrm{B}(\mathrm{x}, (\mathrm{Re}(\mathrm{z}))^{\frac{1}{\mathrm{m}}}|^{-1}$$

for all $\mathrm{x}, \mathrm{y} \in \mathrm{E}$ *and* $\mathrm{z} \in \Sigma(\psi)$.

2. *There is a bounded decreasing function* $\xi : [0, \infty) \to [0, \infty)$ *such that*

$$|\mathbb{K}(\mathrm{t}, \mathrm{x}, \mathrm{y})| \leq |\mathrm{B}(\mathrm{x}, \mathrm{t}^{\frac{1}{\mathrm{m}}})|^{-1} \xi\left(\mathrm{d}(\mathrm{x}, \mathrm{y})^{\mathrm{m}} \mathrm{t}^{-1}\right)$$

for all $\mathrm{x}, \mathrm{y} \in \mathrm{E}$ *and* $\mathrm{t} > 0$.

Proposition (continued)

Then for each $\varepsilon \in (0,1]$ *and* $\theta \in (0, \varepsilon\psi)$, *there is a constant* $\mathrm{C} > 0$ *such that*

$$|\mathbb{K}(\mathrm{z},\mathrm{x},\mathrm{y})| \leq \mathrm{C}|\mathrm{B}(\mathrm{x},(\mathrm{Re}(\mathrm{z}))^{\frac{1}{\mathrm{m}}}|^{-1}\xi\left(\mathrm{d}(\mathrm{x},\mathrm{y})^{\mathrm{m}}|\mathrm{z}|^{-1}\right)^{1-\varepsilon}$$

for all $\mathrm{x},\mathrm{y} \in \mathrm{E}$ *and* $\mathrm{z} \in \Sigma(\theta)$.

We state the following result concerning to domination of semigroups, which is taken from (E. M. Ouhabaz, *Potential Analysis*, 1996).

Proposition

Let $(\mathfrak{a}, \mathrm{D}(\mathfrak{a}))$ *and* $(\mathfrak{b}, \mathrm{D}(\mathfrak{b}))$ *be two positive, symmetric and continuous bilinear forms on* $\mathrm{L}^2(\Omega)$. *Let* T *and* S *be the self-adjoint semigroups on* $\mathrm{L}^2(\Omega)$ *associated with* $\mathfrak{a}$ *and* $\mathfrak{b}$, *respectively. Assume that the semigroups* T *and* S *are positive. Then the following assertions are equivalent.*

Proposition (continued)

(i) *The semigroup* T *is dominated by the semigroup* S *in the sense that*

$$|\mathrm{T}(\mathrm{t})\mathrm{f}| \leq \mathrm{S}(\mathrm{t})|\mathrm{f}|, \quad \textit{for all } \mathrm{f} \in \mathrm{L}^2(\Omega) \textit{ and } \mathrm{t} \geq 0.$$

(ii) $\mathrm{D}(\mathfrak{a})$ *is an ideal in* $\mathrm{D}(\mathfrak{b})$, *in the sense that if* $0 \leq \mathrm{v} \leq \mathrm{u}$ *with* $\mathrm{u} \in \mathrm{D}(\mathfrak{a})$ *and* $\mathrm{v} \in \mathrm{D}(\mathfrak{b})$, *then* $\mathrm{v} \in \mathrm{D}(\mathfrak{a})$, *and*

$$\mathfrak{b}(\mathrm{u}, \mathrm{v}) \leq \mathfrak{a}(\mathrm{u}, \mathrm{v}) \textit{ for all } 0 \leq \mathrm{u}, \mathrm{v} \in \mathrm{D}(\mathfrak{a}).$$

Fractional order Sobolev spaces

Now we introduce the needed fractional order Sobolev spaces. Given $0 < s < 1$ and $\Omega \subset \mathbb{R}^N$ an arbitrary open set whose closure we denote by $\overline{\Omega}$, we set

$$W^{s,2}(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy < \infty \right\},$$

and we endow it with the norm defined by

$$\|u\|_{W^{s,2}(\Omega)} := \left(\int_\Omega |u(x)|^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}}.$$

We set

$$W_0^{s,2}(\overline{\Omega}) := \left\{ u \in W^{s,2}(\mathbb{R}^N) : \; u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

We have the following continuous embeddings (see e.g. Di Nezza-Patalucci-Valdinoci: "*A Hitchhiker's guide to fractional Sobolev spaces*", *Bull. Sci. Math.*, 2012):

- Let $2^\star := \frac{2N}{N-2s}$ if $N > 2s$ and $2^\star \in [2, \infty)$ be arbitrary if $N = 2s$. Then

$$W_0^{s,2}(\overline{\Omega}) \hookrightarrow L^{2^\star}(\Omega). \tag{3.12}$$

- If $N < 2s$, then

$$W_0^{s,2}(\overline{\Omega}) \hookrightarrow C^{0,s-\frac{N}{2}}(\mathbb{R}^N).$$

In this section we state the main results of the article and give some examples. Recall that $\Omega \subset \mathbb{R}^N$ is an arbitrary open set and $T = (T(t))_{t\geq 0}$ is a self-adjoint semigroup on $L^2(\Omega)$ with generator A. Recall also that we say that T has a fractional Gaussian estimate for $0 \leq t \leq 1$ if (3.11) holds and that T has a fractional Gaussian estimate if (3.11) holds for all $t \geq 0$.

The following theorem is the first main result of the paper.

Theorem (4.15)

Assume that T *has a fractional Gaussian estimate for* $0 \leq \mathrm{t} \leq 1$*. Then there exists a constant* $\omega \geq 0$ *such that the semigroup* $(\mathrm{e}^{-\omega \mathrm{t}}\mathrm{T}_{\mathrm{p}}(\mathrm{t}))_{\mathrm{t}\geq 0}$ *is bounded holomorphic of angle* $\frac{\pi}{2}$ *on* $\mathrm{L}^{\mathrm{p}}(\Omega)$ *for every* $1 \leq \mathrm{p} < \infty$*.*

Example 1

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. Let $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ with $D(\mathcal{E}) = u \in W_0^{s,2}(\overline{\Omega}$ be the closed, continuous, non-negative and symmetric bilinear form defined by

$$\mathcal{E}(u,v) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy. \quad (4.1)$$

Let $(-\Delta)_D^s$ be the self-adjoint operator on $L^2(\Omega)$ associated with $\mathcal{E}$ in the sense that

$$D((-\Delta)_D^s) :=$$

$$\Big\{ u|_\Omega : \; u \in W_0^{s,2}(\overline{\Omega}) : \; \exists\, f \in L^2(\Omega);\; \mathcal{E}(u,v) = (f,v)_{L^2(\Omega)}, \quad \forall\, v \in W_0^{s,2}(\overline{\Omega})$$

$$(-\Delta)_D^s(u|_\Omega) = f. \quad (4.2)$$

Example 1 ([continued])

Then $(-\Delta)^s_D$ is the realization in $L^2(\Omega)$ of $(-\Delta)^s$ with the zero Dirichlet exterior condition: $u = 0$ in $\mathbb{R}^N \setminus \Omega$ and is given precisely by

$$\begin{cases} D((-\Delta)^s_D) = \Big\{u|_\Omega : \ u \in W_0^{s,2}(\overline{\Omega}),\ (-\Delta)^s u \in L^2(\Omega)\Big\}, \\ (-\Delta)^s_D(u|_\Omega) = (-\Delta)^s u. \end{cases} \tag{4.3}$$

Example 1 (Continuation)

We have the following results.

1. The operator $-(-\Delta)_{\mathrm{D}}^{\mathrm{s}}$ generates a strongly continuous submarkovian semigroup $(\mathrm{T_s}(\mathrm{t}))_{\mathrm{t}\geq 0}$ on $\mathrm{L}^2(\Omega)$.
2. The semigroup $\mathrm{T_s}$ has a fractional Gaussian estimate.

Example 1 (Continuation)

As a consequence, it follows from Theorem 4.16 that the semigroup $(\mathrm{T}_{s,p}(t))_{t\geq 0}$ is bounded holomorphic of angle $\frac{\pi}{2}$ on $\mathrm{L}^p(\Omega)$ for every $1 \leq p < \infty$.

Remark

Firstly, it follows from the embedding (3.12) *that if* Ω *is bounded, then the embedding* $W_0^{s,2}(\overline{\Omega}) \hookrightarrow L^2(\Omega)$ *is compact and this would imply that the operator* $(-\Delta)_D^s$ *has a compact resolvent and its spectrum consists of eigenvalues* $(\lambda_n)_{n\in\mathbb{N}}$ *satisfying* $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ *and* $\lim_{n\to\infty} \lambda_n = \infty$. *Secondly, let us mention that the operator* $-(-\Delta)_D^s$ *is different from the spectral Dirichlet fractional Laplacian, that is, the (negative) fractional* s*-power of the Laplace operator with zero Dirichlet boundary condition.*

Remark

Their first eigenvalues are different, the eigenfunctions of the spectral Dirichlet fractional Laplacian are smooth (it has the same eigenfunctions as the Dirichlet Laplacian) and this is not the case for $-(-\Delta)^{s}_{D}$ where the eigenfunctions are not even Lipschitz continuous (they are Hölder continuous). For more details we refer to (Servadei-Valdinoci, Proc. Royal Soc. Edinburgh, (2014)) and Biccari-Warma-Zuazua, (Adv. Nonlinear Studies, (2017)).

The proof of the main result uses the following preliminary results.

Lemma

Assume that T *has a fractional Gaussian estimate for* $0 \leq \mathrm{t} \leq 1$*. Then there is a constant* $\omega \geq 0$ *such that the semigroup* $(\mathrm{e}^{-\omega \mathrm{t}}\mathrm{T}(\mathrm{t}))_{\mathrm{t} \geq 0}$ *has a fractional Gaussian estimate for all* $\mathrm{t} \geq 0$*.*

Proof.

By assumption $|T(t)f| \leq Me^{-tb(-\Delta)^s_{\mathbb{R}^N}}|f|$ for $0 \leq t \leq 1$ and $f \in L^2(\Omega)$. Let $t \geq 1$ and write $t = n + \tau$ with $0 \leq \tau < 1$ and $n \in \mathbb{N}$. Then using the semigroup property we get that

$$\begin{aligned} |T(t)f| =& |T(n)T(\tau)f| = |(T(1))^n T(\tau)f| \\ \leq& M^{n+1} e^{-nb(-\Delta)^s_{\mathbb{R}^N}} e^{-\tau b(-\Delta)^s_{\mathbb{R}^N}} |f| \\ \leq& M^{n+1} e^{-tb(-\Delta)^s_{\mathbb{R}^N}} |f| \leq Me^{\omega t} e^{-tb(-\Delta)^s_{\mathbb{R}^N}} |f|, \end{aligned}$$

for every $f \in L^2(\Omega)$, where $\omega = \ln(M)$ and we have assumed that $M \geq 1$. The proof is finished. □

Lemma

Assume that T *has a fractional Gaussian estimate. Then the operator* $\mathrm{T(t)}$ *is given by a kernel* $\mathrm{K}(\mathrm{t},\cdot,\cdot) \in \mathrm{L}^\infty(\Omega\times\Omega)$. *That is,*

$$(\mathrm{T(t)f})(\mathrm{x}) = \int_\Omega \mathrm{K}(\mathrm{t},\mathrm{x},\mathrm{y})\mathrm{f}(\mathrm{y})\,\mathrm{dy}, \quad \forall\, \mathrm{t} > 0,\ \mathrm{f} \in \mathrm{L}^2(\Omega) \text{ and } \mathrm{x}\in\Omega. \tag{4.4}$$

The same is true for $\mathrm{T(z)}$, $\mathrm{z} \in \Sigma(\frac{\pi}{2})$. *Denoting by* $\mathrm{K}(\mathrm{z},\cdot,\cdot)$ *the corresponding kernel of* $\mathrm{T(z)}$, *we have that there are two constants* $\mathrm{b} > 0$ *and* $\mathrm{C} > 0$ *such that*

$$|\mathrm{K}(\mathrm{z},\mathrm{x},\mathrm{y})| \le C(\mathrm{Re(z)})^{-\frac{\mathrm{N}}{2\mathrm{s}}}\left(1 + |\mathrm{x}-\mathrm{y}||\mathrm{bz}]|^{-1/2\mathrm{s}}\right)^{-(\mathrm{N}+2\mathrm{s})(1-\varepsilon)} \tag{4.5}$$

for all $\mathrm{x},\mathrm{y}\in\Omega$ *and* $\mathrm{z}\in\Sigma(\frac{\pi}{2})$.

Proof of Theorem 4.16

Step 1 . Recall that it follows from the first Lemma above that there is a constant $\omega \geq 0$ such that the semigroup $(e^{-\omega t}T(t))_{t\geq 0}$ has a fractional Gaussian estimate. We therefore assume that $\omega = 0$, that is, T has a fractional Gaussian estimate.

Step 2. The family $(T(z))_{Re(z)>0}$ defined with the kernel $K(z, x, y)$ induces a family bounded operators $(T - p(z))_{Re(z)>0}$ on $L^p(\Omega)$ and $T_p(z)f = T(z)f$ for all $f \in L^p(\Omega) \cap L^2(\Omega)$. The semigroup property follows.

Step 3. Holomorphy of family $(\mathrm{T_p(z)})_{\mathrm{Re(z)}>0}$ is proved using equivalence of weak and strong holomorphy and Vitali's theorem.

Step 4. We prove that that $\mathrm{T_p(z)}$ is strongly continuous.

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