Non-perturbative effects and instantons in matrix models and orthogonal polynomials

Albarracín, May 2016

Quantum Gauge field theory and matrix models

Quantum field theories use partition functions *defined* by functional integrals

$$Z = \int e^{-S[A]}\, \mathcal{D}A(x),$$

where $A(x)$ are Gauge fields

$$A(x) \in SU(n) \text{ or } U(n).$$

Expectation values are expressions of the form

$$\langle A(x_1) \cdots A(x_k) \rangle = \frac{1}{Z} \int A(x_1) \cdots A(x_k)\, e^{-S[A]}\, \mathcal{D}A(x)\,.$$

't Hooft question (1974): What happens as $n \to \infty$?

Applications of matrix models

- Quantum Gauge field theory (2000's): Large-n expansions of Hermitian and non-Hermitian matrix models.
- Non-perturbative effects in large-n expansions (2010's): Instantons, oscillatory terms, eigenvalue tunneling, phase transitions, gap closing in eigenvalue distributions...
- Physical Review Letters, Nuclear Physics B, Journal of High Energy Physics,Annals of Physics...

Perturbative and non-perturbative effects

Physical magnitudes are often introduced in terms of divergent perturbative series expansions

$$F(x) \sim \sum_{k \geq 0} \frac{1}{x^k} F_k, \quad x \to +\infty.$$

Non-perturbative corrections are important

$$F(x) \sim \sum_{k \geq 0} \frac{1}{x^k} F_k + \sum_m e^{-A_m x} \, G^{(m)}(x).$$

Instantons: $\mathrm{Re}\, A_m > 0$. Oscillations: $\mathrm{Re}\, A_m = 0$.

MATRIX MODELS AND ORTHOGONAL POLYNOMIALS

Matrix models

Partition function:

$$Z_n = \frac{1}{n!} \int_{\Gamma^n} \prod_{j<k} (z_j - z_k)^2 \exp\left(-\frac{1}{g} \sum_{i=1}^{n} W(z_i) \right) \mathrm{d}^n \mathbf{z},$$

where, in general, Γ is a path in the complex plane and $W(z)$ is a complex-valued function and $g > 0$.

"Physical form":

$$Z_n = \frac{1}{n!} \int_{\Gamma^n} e^{-n^2 S_n(\mathbf{z})} \mathrm{d}^n \mathbf{z},$$

where the discrete action $S_n(\mathbf{z})$ is given by

$$S_n(\mathbf{z}) = \frac{1}{gn^2} \sum_{i=1}^{n} W(z_i) - \frac{1}{n^2} \sum_{j<k} \log(z_j - z_k)^2.$$

Orthogonal polynomials

Heine's formula:

$$P_n(z) = \frac{1}{n!\, Z_n} \int_{\Gamma^n} e^{-n^2 S_n(\mathbf{z})} \prod_{i=1}^{n} (z - z_i)\, \mathrm{d}^n \mathbf{z}.$$

$$\int_{\Gamma} P_n(z) z^k \, e^{-\frac{1}{g} W(z)} \mathrm{d}z = 0, \quad k = 0, \dots, n-1.$$

How to calculate Z_n ?

$$\boxed{Z_n = h_0 h_1 \cdots h_{n-1}}$$

where

$$h_k = \int_\Gamma P_k(z)^2 e^{-\frac{1}{g}W(z)} \mathrm{d}z.$$

If there is a three-term recursion relation

$$zP_k(z) = P_{k+1}(z) + s_k P_k(z) + r_k P_{k-1}(z).$$

then $h_k = r_k h_{k-1}$ and

$$\boxed{Z_n = h_0^n \prod_{k=1}^{n-1} r_k^{n-k}}$$

t' Hooft sequences and large-n limit

$$Z_n = \frac{1}{n!} \int_{\Gamma^n} \prod_{j<k} (z_j - z_k)^2 \exp\left(-\frac{1}{g} \sum_{i=1}^{n} W(z_i) \right) \mathrm{d}^n \mathbf{z},$$

t' Hooft sequences: We take sequences of coupling constants of the form

$$g_n = \frac{t}{n},$$

where t is fixed (t' Hooft parameter).

The free energy

- Main objects: the free energy:

$$F_n = -\frac{1}{n^2} \log |Z_n(g_n)|.$$

  and its planar limit $F = \lim_{n\to\infty} F_n$.

- Physical magnitudes:

$$\text{Entropy} = F'(t), \quad \text{Specific heat} = F''(t).$$

- Phases of the model = Analyticity regions of $F(t)$
- Critical points= Singularities t_c of $F(t)$.

Basic questions

1. Existence of F.
2. Existence of large-n asymptotic expansions of F_n.

Saddle point method and eigenvalue density

The saddle points of the action $S_n(\mathbf{z})$ are the solutions $\mathbf{z}^{(n)} = (z_1, \ldots, z_n)$ of

$$\frac{1}{g} W'(z_i) = \sum_{j \neq i} \frac{2}{z_i - z_j},$$

Requirements

As $n \to \infty$ there exists a condensate of saddle points with a unit normalized density $\rho(z)$ (eigenvalue density)

$$\frac{1}{n} \sum_{i=1}^{n} \delta(z - z_i^{(n)}) \to \rho(z)\, |\mathrm{d}z|.$$

with a support γ given by one or several connected arcs (CUTS)

$$\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \gamma_m$$

1. Each cut is close to a critical point of the potential $W(z)$. The number and form of the cuts depend on t.
2. The large-n limit of $S_n(\mathbf{z})$ is given by the functional

$$S[\rho] = \frac{1}{t}\int_\gamma W(z)\rho(z)|\mathrm{d}z| - \frac{1}{2}\int_\gamma |\mathrm{d}z| \int_\gamma |\mathrm{d}z'| \log(z-z')^2 \rho(z)\rho(z').$$

and satifies

$$Z_n = \exp\Big(-n^2 S[\rho] + o(n^2)\Big), \quad \text{as } n \to \infty.$$

Hence the planar free energy $F = \mathrm{Re}\, S[\rho]$ is given by the total electrostatic energy E

$$F = E = \frac{1}{t}\int_\gamma V(z)\rho(z)|\mathrm{d}z| - \int_\gamma |\mathrm{d}z| \int_\gamma |\mathrm{d}z'| \log|z-z'|\rho(z)\rho(z'),$$

where $V(z) = \mathrm{Re}\, W(z)$ (Bessis-Itzykson-Zuber (BIZ) formula).

How to calculate $\rho(z)$?

Polynomials associated to saddle points $(z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)})$:

$$p_n(z) = \prod_i (z - z_i^{(n)}).$$

The saddle point equations imply the the second-order differential equation

$$\boxed{p_n''(z) - \tfrac{1}{g_n} W'(z)\, p_n'(z) + F(z)\, p_n(z) = 0}$$

where

$$F(z) = -\frac{1}{g_n} \sum_{i=1}^{n} \frac{W'(z) - W'(z_i^{(n)})}{z - z_i^{(n)}}.$$

In the t' Hooft limit

$$\frac{1}{n}\frac{p_n'(z)}{p_n(z)} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{z - z_i^{(n)}} \to \omega(z) = \int_\gamma \frac{\rho(z')|\mathrm{d}z'|}{z - z'}.$$

and it follows that

$$\boxed{\left(\frac{1}{t}\,W'(z) - 2\,\omega(z)\right)^2 = R(z),}$$

where $R(z)$ is a rational function for potentials $W(z)$ with rational derivative $W'(z)$. This is the main tool to determine $\rho(z)$.

Hermitian models

For $\Gamma = \mathbb{R}$ and real analytic potentials $W(x)$ such that

$$\lim_{|x|\to\infty} \frac{W(x)}{\log|x|} = \infty,$$

- The eigenvalue density exists and

$$\begin{aligned} \rho(z) &= \text{equilibrium density of } E[\rho] \text{ on } \mathbb{R} \\ &= \text{asymptotic zero density of } P_n(z). \end{aligned}$$

- The planar free energy F exists and $F = E$.

For Hermitian models in the ONE-CUT case there exists a perturbative series expansion

$$F_n - F_n^{(Gauss)} \sim F^{(0)} + \frac{1}{n^2} F^{(1)} + \cdots + \frac{1}{n^{2k}} F^{(k)} + \cdots .$$

where $F_n^{(Gauss)}$ is the free energy for the Hermitian Gaussian model $W(z) = z^2/2$.

- Multi-cut case ?
- Non-Hermitian models ?

NON-PERTURBATIVE TERMS in LARGE-n EXPANSIONS

Oscillatory terms

They arise in the large-n expansions of Hermitian models (equiilbrium densities on $\mathbb{R}$).

- In the two-cut case $\gamma = [a_1, a_2] \cup [a_3, a_4]$ $(a_1 < a_2 < a_3 < a_4)$

  $$F_n - F_n^{(Gauss)} \sim F^{(0)} + \frac{1}{n^2} \log \theta_3(n\, x) + \frac{1}{n^2} F^{(1)} + \cdots ,$$

  where θ_3 is the Jacobi elliptic function.

  $$\theta_3(z) = \sum_{k=-\infty}^{\infty} q^{k^2/2} e^{2\pi i k z}, , \quad q = \exp\Big(- \pi^2 / E'' \Big).$$

  and x is the charge contained in the cut $[a_3, a_4]$.

- Formula derived by Bonnet-David-Eynard (2000) and proved by Claeys-Grava-McLaughlin (2015, 75 pages) for polynomial $W(z)$

  .

Instanton terms

- Heuristic formulas derived by Mariño, Schiappa, Weiss,... (2006) for general multi-cut models.
- For example, in the two-cut case $\gamma = \gamma_1 \cup \gamma_2$ they find a non-perturbative term of the form

$$F_n - F_n^{(Gauss)} \sim F^{(0)} + \frac{1}{n^2} \log \theta_3\left(\frac{nA}{2\pi i}\right) + \frac{1}{n^2} F^{(1)} + \cdots$$

Using Jacobi's triple identity this term leads to a sum of instanton contributions to F_n of the form

$$\frac{1}{n^2} \sum_{m \neq 0} e^{-mnA} \frac{(-1)^m}{m(q^{m/2} - q^{-m/2})}.$$

The constant A is the potential barrier $A = U\Big|_{\gamma_2} - U\Big|_{\gamma_1}$ between the cuts (Instanton action), where the total potential

$$U(z) = \frac{1}{t}V(z) - 2\int_{\gamma_l} \ln|z - z'|\rho(z')|\mathrm{d}z'|,$$

is constant on each cut of the support of $\rho(z)$:

$$U\Big|_{\gamma_i} = U_i, \quad i = 1, 2.$$

Physical interpretation

The instanton terms represent contributions to the partition function

$$Z_n = \frac{1}{n!}\int_{\Gamma^n} e^{-n^2 S_n(\mathbf{z})}\mathrm{d}^n\mathbf{z},$$

corresponding to extremal densities ρ_e of the action $S[\rho]$ in which eigenvalue tunneling takes place.

Many doubts on the properties assumed for non-Hermitian models

Our aim (with G. Alvarez and E.Medina):

- To investigate the large-n limit for exact non-Hermitian models
- To use rigorous results from the theory of non-Hermitian orthogonal polynomials : critical densities, S-property...

TWO EXACT MODELS

The Penner (Laguerre) model:

$$W(z) = z + \log z.$$

The two-Penner (Jacobi) model:

$$W(z) = -\mu_+ \log(1 - z) - \mu_- \log(z + 1).$$

THE BARNES G FUNCTION

The Barnes function is defined by the canonical product

$$G(1+z) = (2\pi)^{z/2} e^{-\frac{1}{2}(z+z^2(1+\gamma))} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right)^k e^{-z+z^2/2k}. \qquad (1)$$

It satisfies

$$G(1+z) = \Gamma(z)G(z), \quad G(1) = 1.$$

Main identity:

$$\prod_{k=1}^{n-1} (k+\alpha)^{n-k} = \frac{G(1+n+\alpha)}{G(1+\alpha)\Gamma(1+\alpha)^n}.$$

Asymptotics of the Barnes function

As $x \to \infty$

$$\ln G(1+x) \sim \frac{x^2}{2}\ln x - \frac{3}{4}x^2 + \frac{x}{2}\ln(2\pi) - \frac{\ln x}{12} + \zeta'(-1) + \varphi(x).$$

Here $\varphi(x)$ is the divergent series

$$\varphi(x) = \sum_{m=2}^{\infty} \frac{B_{2m}}{2m(2m-2)} \frac{1}{x^{2m-2}}.$$

which is the genus expansion of the free energy of a one-dimensional string theory on a circle.

As $x \to \infty$ two additional oscillatory terms appear

$$\ln|G(1-x)| - x\ln\left|\frac{\sin(\pi x)}{\pi}\right| - \frac{1}{2\pi}\mathrm{Cl}_2(2\pi x) \sim$$

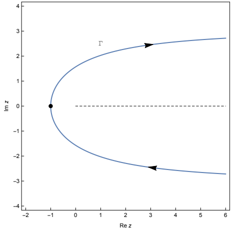
$$\frac{x^2}{2}\ln x - \frac{3}{4}x^2 + \frac{x}{2}\ln(2\pi) - \frac{\ln x}{12} + \zeta'(-1) + \varphi(x).$$

where $\mathrm{Cl}_2(x)$ is the Clausen function

$$\mathrm{Cl}_2(x) = \sum_{m=1}^{\infty} \frac{\sin(mx)}{m^2}.$$

THE PENNER MODEL : $W(z) = z + \log z.$

The path Γ encircles the positive real axis



The associated orthogonal polynomials are proportional to the Laguerre polynomials $L_n^{(\alpha)}(z/g)$ with $\alpha = -\frac{1}{g}$. The recursion coefficient is

$$r_k = k\, g(kg - 1).$$

RESULTS

The partition function is given by

$$Z_n = g^{n(n-\frac{1}{g})} \left(1 - e^{-i\frac{2\pi}{g}}\right)^n \frac{G(1+n)G\left(1+n-\frac{1}{g}\right)}{G\left(1-\frac{1}{g}\right)}.$$

The large-n expansion of F_n can be derived from the asymptotic expansions of the Barnes function.

The saddle-points of the action are the zeros of

$$L_n^{(-1/g_n)}\left(\frac{z}{g_n}\right).$$

We may use the results on the asymptotic zero distribution of Laguerre polynomials obtained by

Martínez Finkelstein, Martínez Gonzalez, Orive 2001;
Kuijlaars- McLaughlin 2004: .
Diaz Mendoza, Orive 2011.

Phases of the Penner model

Weak-coupling phase $0 < t < 1$

Large n expansion for 't Hooft sequences $g_n = t/n$

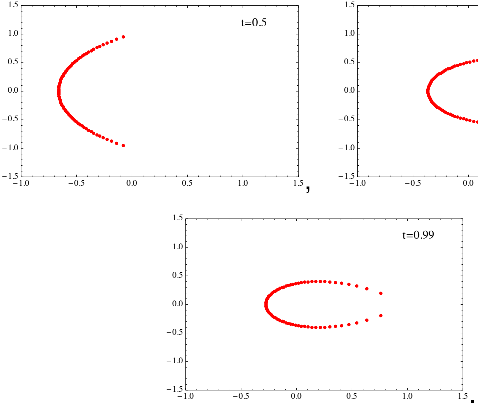
$$F_n \approx \left(-\frac{1}{2}\log t + \frac{3}{2}\left(\frac{t-1}{t}\right) - \frac{1}{2}\left(\frac{t-1}{t}\right)^2 \log|t-1|\right)$$
$$-\frac{1}{n}\ln(2\pi) + \frac{1}{12n^2}\ln n - \frac{1}{n^2}\left(\zeta'(-1) - \frac{1}{12}\ln(1-t)\right)$$
$$-\sum_{k=2}^{\infty}\frac{B_{2k}}{2k(2k-2)n^{2k}}\left(1 + t^{2k-2}\left((1-t)^{2-2k} - 1\right)\right).$$

The planar free energy exists and is given by

$$F = E = -\frac{1}{2}\log t + \frac{3}{2}\left(\frac{t-1}{t}\right) - \frac{1}{2}\left(\frac{t-1}{t}\right)^2 \log|t-1|.$$

The saddle points condensate on one-cut which closes as $t \to 1$

Gap closing



Strong-coupling phase $1 < t < +\infty$

Large n expansion for 't Hooft sequences $g_n = t/n$:

$$
\begin{aligned}
F_n \approx\ & \frac{1}{n}\left(\frac{1}{t}-1\right)\ln\left|\sin\left(\frac{\pi n}{t}\right)\right| + \frac{1}{2\pi n^2} Cl\left(\frac{2\pi n}{t}\right) \\
& + \left(-\frac{1}{2}\log t + \frac{3}{2}\left(\frac{t-1}{t}\right) - \frac{1}{2}\left(\frac{t-1}{t}\right)^2 \log|t-1|\right) \\
& - \frac{1}{n}\ln(2\pi) + \frac{1}{12n^2}\ln n - \frac{1}{n^2}\left(\zeta'(-1) - \frac{1}{12}\ln(t-1)\right) \\
& - \sum_{k=2}^{\infty} \frac{B_{2k}}{2k(2k-2)n^{2k}} \left(1 + t^{2k-2}\left((t-1)^{2-2k} - 1\right)\right).
\end{aligned}
$$

Then the planar free energy does not exist.

Kuijlaars-McLaughlin (KM) sequences

However, if we use sequences g_n such that

$$lim_{n\to\infty} g_n n = t \quad \text{and} \quad \exists \quad L = \lim_{n\to\infty} |\sin(\pi/g_n)|^{1/n} \ ,$$

They lead to a term in the large-n expansion of the form

$$\left(1 - \frac{1}{n g_n}\right) \log \left| \sin\left(\frac{\pi}{g_n}\right) \right|^{1/n},$$

and then the planar free energy does exist and is given by

$$F = E + \left(\frac{1}{t} - 1\right) \log L,$$

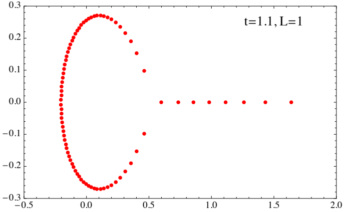
where the total energy E is

$$E = -\frac{1}{2}\log t + \frac{3}{2}\left(\frac{t-1}{t}\right) - \frac{1}{2}\left(\frac{t-1}{t}\right)^2 \log(t-1).$$

The eigenvalue density depends not only on the value of the 't Hooft parameter t but also on L. Its support is of the form

$$\gamma_L = C_L \cup [a, b],$$

where C_L is a closed loop. For $L \neq 1$ it is a two-cut support



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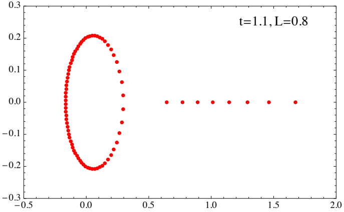


The total potential is constant on the cuts

$$U\Big|_{[a,b]} = \frac{1}{t}\left[2 - \frac{1}{t} - \log t - \left(1 - \frac{1}{t}\right)\log(t-1)\right],$$

with a potential barrier

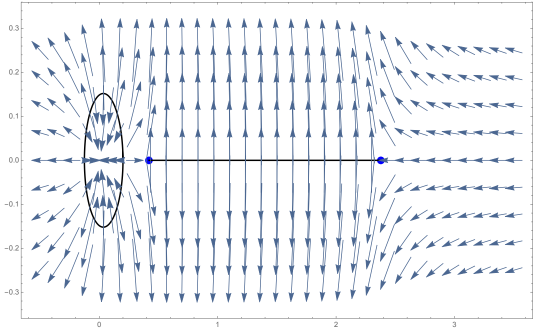
$$U\Big|_{C_L} - U\Big|_{[a,b]} = -\log L.$$

This leads to an instanton action for $L \neq 1$ given by

$$A = -\log L.$$

However, there are not instanton terms but oscillatory terms in the expansion of F_n .

S-property:



Phase transitions with KM sequences

- At $t = 1$ there is a first-order phase transition at $t = 1$ for $L \neq 1$

$$\Delta F = 0, \quad \Delta F' = \log L.$$

- At $t = 1$ there is a continuous phase transition for $L = 1$.

$$\Delta F = \Delta F' = 0, \quad \lim_{t \to 1} F'' = \infty.$$

THE TWO-PENNER MODEL :

$$W(z) = -\mu_+ \log(1 - z) - \mu_- \log(z + 1),$$

Closely related to the Jacobi polinomials $P_n^{(\alpha,\beta)}(z)$:
with

$$\alpha = \frac{\mu_+}{g}, \quad \beta = \frac{\mu_-}{g}.$$

Recurrence coefficient

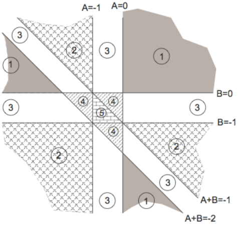
$$r_k = \frac{4k(k+\alpha)(k+\beta)(k+\alpha+\beta)}{(2k+\alpha+\beta)^2(2k+1+\alpha+\beta)(2k-1+\alpha+\beta)}.$$

Classical case: $\alpha, \beta > -1$. Hermitian orthogonality on $[-1, 1]$.

We consider large-n limits with $ng_n \to t > 0$. If we denote

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = \frac{\mu_+}{t} \equiv A, \quad \lim_{n\to\infty} \frac{\beta_n}{n} = \frac{\mu_-}{t} \equiv B.$$

There are five cases:
Martínez Finkelstein, Orive, Kuijlaars (2004, 2005)



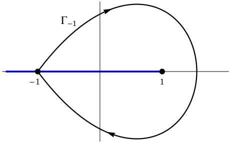
There are three non-classical cases such that the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ are orthogonal on a single path for appropriate conditions on α and β.

RESULTS

If we take

$$\alpha < -1, \quad \beta > -1,$$

then the Jacobi polynomials are orthogonal on the integration path



We consider large-n limits with

$$-1 < A < 0 < B,$$

or, equivalently,

$$\mu_+ < 0, \quad \mu_- > 0, \quad t > |\mu_+|,$$

The planar free energy

The partition function is given by

$$|Z_n| = |\sin(\pi\alpha)|^n \frac{G(1+n)G(1+n+\alpha)G(1+n+\beta)G(1+n+\alpha+\beta)}{G(1+\alpha)G(1+\beta)G(1+2n+\alpha+\beta)}.$$

For t' Hooft sequences $g_n = t/n$ the planar free energy F does not exist.

The eigenvalue density

The saddle points of the action S_n are the zeros of $P_n^{(\alpha_n - 1, \beta_n - 1)}(z)$.

We use the results on the asymptotic zero distribution of Jacobi polynomials obtained by

Martínez Finkelstein, Orive (2005).

Kuijlaars-McLaughlin (KM) sequences

For KM sequences g_n

$$lim_{n\to\infty} g_n n = t \quad \text{and} \quad \exists \quad L = \lim_{n\to\infty} |\sin(\pi A n)|^{1/n}$$

the planar free energy F and the eigenvalue density $\rho(z)$ exist and

$$F = E - \Big(1 + A\Big) \log L.$$

where the total energy is

$$E = -(1 + A + B)\ln 2 + \frac{A^2}{2}\ln|A| + \frac{B^2}{2}\ln B$$

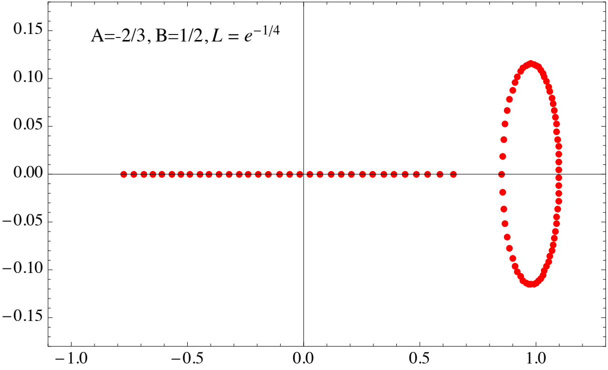
$$-\frac{(1+A)^2}{2}\ln(1+A) - \frac{(1+B)^2}{2}\ln(1+B)$$

$$-\frac{(1+A+B)^2}{2}\ln(1+A+B) + \frac{(2+A+B)^2}{2}\ln(2+A+B).$$

For KM sequences the support of the eigenvalue density is of the form

$$\gamma_L = C_L \cup [a, b],$$

where C_L is a closed loop. For $L \neq 1$ it is a two-cut support.



The total potential is z-independent on the cuts with a potential barrier

$$U\Big|_{C_L} - U\Big|_{[a,b]} = \log L$$

such that

$$U\Big|_{[a,b]} = -(2+A+B)\ln 2 - (1+A)\ln 1 + A) - (1+B)\ln(1+B)$$

$$-(1+A+B)\ln(1+A+B) + 2(2+A+B)\ln(2+A+B),$$

This leads to an instanton action for $L \neq 1$ given by

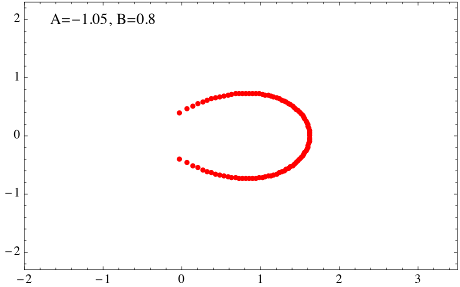
$$A = \log L.$$

However, there are not instanton terms but oscillatory terms in the expansion of F_n .

Phase transitions for KM sequences

There is a rich structure of phase transitions depending on the values of the parameters $\mu_{\pm}$ and t. For example

Gap closing as $A \to -1$ with $A < -1,\ B > 0$



SOME CONCLUSIONS

For non-hermitian matrix models:

- Due to the possible presence of non-perturbative oscillating terms, 't Hooft sequences do not always lead to a well-defined planar free energy F .
- Sequences different from 't Hooft sequences with a well-defined planar free energy are worth considering.
- Instanton effects in large-n expansions are not always associated with potential barriers.

Rigorous results on non-hermitian orthogonal polynomials should be useful to understand non-perturbative phenomena in non-hermitian matrix models.

OTHER INTERESTING MODELS

Multi-Penner models

$$W(z) = \sum_{i=1}^{k} \rho_i \log(z - q_i).$$

String models and conformal Toda field theories. Heine-Stieltjes polynomials.

The Gross-Witten-Wadia (GWW) model

- Introduced in 1980.
- It describes 2D Quantum Gauge theory on a lattice.
- For 't Hooft sequences it reduces to a modified Penner model

$$W(z) = t \log z - \left(z + \frac{1}{z}\right),$$

  on the unit circle $\Gamma = C(0; 1)$.

- Recently (Phys. Rev. Letters, April 2016) Buividovich, Dune and Valgushev have found numerically a phase transition driven by a gap closing and a cut birth at $t = 2$.