Fractional Derivatives

Define **a** fractional integral of complex order z by

$$(W^{-z}f)(x) := \frac{1}{\Gamma(z)} \int_x^\infty (t-x)^{z-1} f(t)\, dt, \quad \operatorname{Re} z > 0,$$

and **a** fractional derivative of complex order z by

$$(W^{z}f)(x) := \frac{1}{\Gamma(n-z)} \frac{d^n}{dx^n} \int_x^\infty (t-x)^{n-z-1} f(t)\, dt, \quad n < \operatorname{Re} z < n+1.$$

$f \in \mathcal{S}(\mathbb{R}^+)$, the Schwartz space of functions supported on $[0,\infty)$.

Difference Operator of Complex Order

Backward difference operator: Let $g \in C(\mathbb{R})$.

$$\nabla g(x) := g(x) - g(x-1),$$
$$\nabla^n g(x) := \nabla(\nabla^{n-1} g)(x) = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} g(x-k), \quad 1 < n \in \mathbb{N}.$$

Backward difference operator of complex order $z \in \mathbb{C}_{>0}$:

$$\nabla^z g(x) := \sum_{k=0}^{\infty} (-1)^k \binom{z}{k} g(x-k), \qquad g \in \mathcal{S}(\mathbb{R}).$$

Complex Dividided Differences

Finite sequence of knots: $t_0 < t_1 < \ldots < t_n$ on $\mathbb{R}$.

n-th order dividided difference for a function $g \in C(\mathbb{R})$:

$$\begin{aligned} [t_0]g &:= g(t_0) \quad \text{for } n = 0, \\ [t_0, \ldots, t_n]g &:= \frac{[t_0, \ldots, t_{n-1}]g - [t_1, \ldots, t_n]g}{t_0 - t_n} \quad \text{for } n \geq 1. \end{aligned}$$

Non-recursive equivalent:

$$[t_0, \ldots, t_n]g = \sum_{j=0}^{n} \frac{g(t_j)}{\prod_{l \neq j}(t_j - t_l)}.$$

Define a complex dividided difference operator of order $z \in \mathbb{C}_+$ for uniformly distributed knots $\{0, 1, 2, \ldots\} =: \mathbb{N}_0$ by:

$$[z; \mathbb{N}_0]g := \sum_{k=0}^{\infty} (-1)^k \frac{g(k)}{\Gamma(z-k+1)\Gamma(k+1)}, \quad g \in \mathcal{S}(\mathbb{R}).$$

Complex B-Splines and Difference Operators

Theorem (Forster, M. 2009)

Let $z \in \mathbb{C}_{>1}$ *and let* $g \in \mathcal{S}(\mathbb{R}^+)$. *Then*

1. $\nabla^z g(0) = \int_{\mathbb{R}} B_z(t)(D^z g)(t)\,dt.$

2. $B_z(x) = z[z;\mathbb{N}_0](\bullet - k)_+^{z-1}.$

3. $[z;\mathbb{N}_0]g = \dfrac{1}{\Gamma(z)} \int_{\mathbb{R}} B_z(t)(D^z g)(t)\,dt.$

(B. Forster, P. M.: Statistical Encounters with Complex B-Splines, Const. Approx. 29, (2009), 325–344)

Weak Integral Formulation

Interpret $\int_{\mathbb{R}} B_z(t)g(t)dt$ as a weak integral:

$$\int_{\mathbb{R}} B_z(t)g(t)dt =: \left\langle \int_{\mathbb{R}} B_z(t) \bullet dt, g \right\rangle.$$

Then

$$\begin{aligned}
\left\langle \int_{\mathbb{R}} B_z(t) \bullet dt, g \right\rangle &= \int_{\mathbb{R}} B_z(t)g(t)dt \\
&= \int_{\mathbb{R}} \sum_{k\geq 0} (-1)^k \binom{z}{k} \frac{1}{\Gamma(z)} (t-k)_+^{z-1} g(t)dt \\
&= \sum_{k\geq 0} (-1)^k \binom{z}{k} (W^{-z}g)(k) \\
&= (\Delta^z W^{-z} g)(0) = \left\langle \delta, \Delta^z W^{-z} g \right\rangle,
\end{aligned}$$

where δ denotes the Delta distribution.

Distributional Formulation

Replace g by $g^{(z)} = W^z g \in \mathcal{S}(\mathbb{R}^+)$:

$$\left\langle \int_{\mathbb{R}} B_z(t) \ \bullet \ dt, g^{(z)} \right\rangle = \langle \delta, \Delta^z g \rangle .$$

Hence, $\langle \int_{\mathbb{R}} B_z(t) \ \bullet \ dt, \ \cdot \ \rangle$ can be interpreted as a fractional integration operator of order z.

Also:

$$B_z \longmapsto \underbrace{\left(g \longmapsto \int\limits_{\mathbb{R}} B_z(t) g(t) dt \right)}_{\in \mathcal{S}'(\mathbb{R})}, \quad \forall \, g \in \mathcal{S}.$$

Choosing the Correct Function Space

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$, $n \in \mathbb{N}$.

Denote by $\Delta : \mathcal{S} \to \mathcal{S}$, $\Delta := -\sum_{j=1}^{n} \partial_j^2$, the Laplace operator on $\mathcal{S}$.

The Lizorkin spaces Φ and Ψ are defined by

$$\Phi := \bigcap_{m \in \mathbb{N}} \Delta^m(\mathcal{S}),$$

and

$$\Psi := \left\{ \psi \in \mathcal{S} : \partial^\mu \psi(\mathbf{0}) = 0, \forall \mu \in \mathbb{N}^n \right\},$$

respectively.

Ψ is a closed ideal of $\mathcal{S}$.

Relation between Φ and Ψ

For all $\psi \in \mathcal{S}$ are equivalent:

(i) $\psi \in \Psi$;

(ii) $(\partial^{\mu}\psi)(\boldsymbol{\xi}) \in o(\|\boldsymbol{\xi}\|^{t})$ as $\|\boldsymbol{\xi}\| \to 0$, $\forall \mu \in \mathbb{N}^{n}$, $\forall t \in \mathbb{R}^{+}$;

(iii) $\|\boldsymbol{\xi}\|^{-2m}\,\psi \in \mathcal{S}$, $\forall m \in \mathbb{N}$.

The following relationship holds between Φ and Ψ:

$$\mathcal{F}(\Phi) \cong \Psi,$$

where $\mathcal{F}$ denotes the Fourier transform.

For any $\phi \in \mathcal{S}$:

$$\phi \in \Phi \iff \langle p, \phi \rangle = 0, \quad \text{for any polynomial } p.$$

Relation between Φ', Ψ', and $\mathcal{S}'$

Let $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n)$ be the topological dual of $\mathcal{S}$. Then the topological duals of Φ and Ψ are as follows.

(i) $\Phi' = \mathcal{S}'/\mathcal{P}$.

(ii) $\Psi' = \mathcal{S}'/\mathcal{F}(\mathcal{P})$,

where

$$\mathcal{P} := \{f \in \mathcal{S}' \mid \operatorname{supp} \mathcal{F}(f) = \{0\}\}$$

denotes the set of polynomials in $\mathcal{S}'$.

Lizorkin Spaces for $n = 1$

Consider the Lizorkin spaces

$$\Psi := \{\psi \in \mathcal{S}(\mathbb{R}) \mid D^m\psi(0) = 0, \forall m \in \mathbb{N}\}.$$

$$\Psi_+ := \{f \in \Psi \mid \operatorname{supp} f \subseteq [0, \infty)\}.$$

Let $z \in \mathbb{C}_+$ and define $K_z : \mathbb{R} \to \mathbb{C}$ by $x \longmapsto \frac{x_+^{z-1}}{\Gamma(z)}$.

For an $f \in \Psi_+$, define a fractional derivative operator $\mathcal{D}^z$ of complex order z on Ψ_+ by

$$\mathcal{D}^z f := \underbrace{(D^k f) * K_{k-z}}_{\text{(Caputo)}} = \underbrace{D^k(f * K_{k-z})}_{\text{(Riemann–Liouville)}}, \qquad k = \lceil \operatorname{Re} z \rceil.$$

Fractional Derivative Operator on Ψ'_+

Note that for $\operatorname{Re} z > -1$, $K_z \in L^1_{\text{loc}}$.

Thus, consider K_z as an element of Ψ'_+.

$$\langle K_z, \varphi \rangle = \int_0^\infty K_z(t)\varphi(t)dt, \quad \forall \varphi \in \Psi_+.$$

Here, $\langle \bullet, \bullet \rangle$ denotes the canonical pairing between Ψ_+ and Ψ'_+.

For $f, g \in \Psi'_+$ the convolution exists on Ψ'_+ and is defined in the usual way by

$$\langle f * g, \varphi \rangle := \langle (f \times g)(x, y), \varphi(x+y) \rangle = \langle f(x), \langle g(y), \varphi(x+y) \rangle \rangle, \ \ \varphi \in \Psi_+.$$

The pair $(\Psi'_+, *)$ is a convolution algebra with the Dirac delta distribution δ as its unit element.

Extend the operators $\mathcal{D}^{\pm z}$ to Ψ'_+ in the following way.

Let $z \in \mathbb{C}_+$, let $\varphi \in \Psi_+$ be a test function, and let $f \in \Psi'_+$.

Then the fractional derivative operator $\mathcal{D}^z$ on Ψ'_+ is defined by

$$\langle \mathcal{D}^z f, \varphi \rangle := \langle (D^k f) * K_{k-z}, \varphi \rangle, \quad k = \lceil \operatorname{Re} z \rceil.$$

and the fractional integral operator $\mathcal{D}^{-z}$ by

$$\langle \mathcal{D}^{-z} f, \varphi \rangle := \langle f * K_z, \varphi \rangle.$$

It is known that

$$\mathcal{D}^z \left[\frac{(x-k)_+^{z-1}}{\Gamma(z)} \right] = \delta(x-k), \quad k < x \in [0, \infty).$$

Thus, by the semi-group properties of $\mathcal{D}^z$

$$\mathcal{D}^{-z} \delta(\bullet - k) = \frac{(\bullet - k)_+^{z-1}}{\Gamma(z)}.$$

Splines of Complex Order

Let $z \in \mathbb{C}_+$ and let $\{a_k \mid k \in \mathbb{N}_0\} \in \ell^\infty(\mathbb{R})$. A solution of the equation

$$\mathcal{D}^z f = \sum_{k=0}^{\infty} a_k \, \delta(\bullet - k), \qquad (*)$$

is called a *spline of complex order* z.

The complex B-spline

$$B_z(x) = \frac{1}{\Gamma(z)} \sum_{k=0}^{\infty} (-1)^k \binom{z}{k} (x-k)_+^{z-1}, \quad \operatorname{Re} z \geq 1.$$

is a solution of Equation $(*)$.

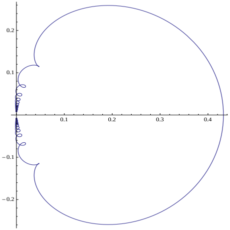
B. Forster, P. M.: Splines of Complex order: Fourier, Filters, and Fractional Derivatives, Sampling Theory in Signal and Image Processing, Vol. 10, No. 1-2 (2011), 89 - 109.

Exponential B-Splines of Complex Order

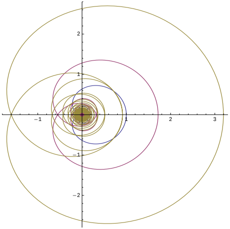
Let $\operatorname{Re} z > 1$.

$$\widehat{B}_z(\omega) = \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^z \quad \longmapsto \quad \widehat{E}_z^a(\omega, a) := \underbrace{\left(\frac{1 - e^{-(a+i\omega)}}{a + i\omega}\right)^z}_{=: \Omega(\omega,a)^z}$$

$\Omega(\omega, a)$ only well-defined for $a \geq 0$:



$a > 0$



$a < 0$

Time-Domain Representation

Theorem (M. 2014)

Let $\operatorname{Re} z > 1$. *Denote the exponential difference operator of complex order* z *by*

$$\nabla_a^z f(x) := \sum_{\ell=0}^{\infty} \binom{z}{\ell} (-1)^\ell e^{-\ell a} f(x-\ell).$$

Then the time domain representation for an exponential B-Splines of complex order z *is given by*

$$E_a^z(x) = \frac{1}{\Gamma(z)} \sum_{\ell=0}^{\infty} \binom{z}{\ell} (-1)^\ell e^{-\ell a} e_+^{-a(x-\ell)} \, (x-\ell)_+^{z-1},$$

where $e_+^{(\bullet)} := \chi_{[0,\infty)} \, e^{(\bullet)}$.

P. M.: Exponential Splines of Complex Order, Contemp. Math. Vol. 626 (2014), 87 - 106

Properties of Exponential B-Splines

Let $a > 0$ and $\operatorname{Re} z > 1$.

- $E_z^a \in \Psi'_+$.
- $\widehat{E}_z^a(\omega) = \widehat{E}_{\operatorname{Re} z}^a(\omega)\, e^{\operatorname{Im} z \arg \Omega(\omega,a)}\, e^{-i \operatorname{Im} z \ln |\Omega(\omega,a)|}$.
- $E_z^0 = B_z$.
- **Define** an operator $(\mathcal{D} + aI)^z : \Psi'_+ \to \Psi'_+$, by

$$(\mathcal{D} + aI)^z (e^{-a(\bullet)} f) := e^{-a(\bullet)} \mathcal{D}^z f,$$

then

(i) $f \equiv 1 \in \Psi'_+ \quad \Longrightarrow \quad e^{-a(\bullet)} \in \ker(\mathcal{D} + aI)^z$.

(ii) $(\bullet)^{z-1} \in \Psi'_+ \quad \Longrightarrow \quad (\bullet)^{z-1}\, e^{-a(\bullet)} \in \ker(\mathcal{D} + aI)^z$.

Exponential Splines of Complex Order

Note that

$$(\mathcal{D} + aI)^z E_z^a = \sum_{\ell=0}^{\infty} \left[\binom{z}{\ell} (-1)^\ell e^{-\ell a} \right] \delta(\bullet - \ell).$$

Let $\operatorname{Re} z > 1$ and let $a > 0$.

An *exponential spline of complex order* is any solution of the (distributional) differential equation

$$(\mathcal{D} + aI)^z f = \sum_{\ell=0}^{\infty} c_\ell \, \delta(\bullet - \ell),$$

for some ℓ^∞-sequence $\{c_\ell \mid \ell \in \mathbb{N}\}$.

P. M.: Exponential Splines of Complex Order, Contemp. Math. Vol. 626 (2014), 87 - 106

Motivation for Extension beyond $\mathbb{C}$

- For certain types of applications such as geophysical data processing a multi-channel description is required. For instance, seismic data has four channels: P (Compression), S (Shear), L (Love) and R (Rayleigh) waves.
- The color value of a pixel in a colour image is composed of three components – the red, green and blue channels.
- The basis functions should have the same analytic properties of fractional/complex B-splines but should in addition be able to describe multi-channel structures.
- A multiresolution structure for the construction of wavelets on the plane for the analysis of four-channel signals was outlined in J. Hogan & A. Morris, Quaternionic wavelets, *Numer. Funct. Anal. Optim.*, **33**(7-9), 2012, 1095–1111..

Quaternions

The real, associative algebra of quaternions $\mathbb{H}_{\mathbb{R}}$ is given by

$$\mathbb{H}_{\mathbb{R}} = \left\{ a + \sum_{i=1}^{3} v_i e_i : a, v_1, v_2, v_3 \in \mathbb{R} \right\},$$

where the imaginary units e_1, e_2, e_3 satisfy

$$e_1^2 = e_2^2 = e_3^2 = -1,$$

and

$$e_1 e_2 = e_3, \quad e_2 e_3 = e_1, \quad e_3 e_1 = e_2.$$

Hence, $\mathbb{H}_{\mathbb{R}}$ is a non-commutative algebra.

Each quaternion $q = a + \sum_{i=1}^{3} v_i e_i$ may be decomposed as

$$q = \operatorname{Sc} q + \operatorname{Ve} q,$$

where $\operatorname{Sc} q = a$ is the *scalar part* of q and $\operatorname{Ve} q = v = \sum_{i=1}^{3} v_i e_i$ is the *vector part* of q.

The *conjugate* $\overline{q}$ of the real quaternion $q = a + v$ is the quaternion $\overline{q} = a - v$.

Note that $q\overline{q} = \overline{q}q = |q|^2 = a^2 + |v|^2 = a^2 + \sum_{i=1}^{3} v_i^2$.

$$\mathbb{H}_{\mathbb{C}} = \left\{ a + \sum_{i=1}^{3} v_i e_i : a, v_1, v_2, v_3 \in \mathbb{C} \right\}. \qquad \text{(Biquaternions)}$$

Quaternionic B-Splines

$\widehat{B_q} : \mathbb{R} \to \mathbb{H}_\mathbb{C}$:

$$\widehat{B_q}(\xi) := \left(\frac{1 - e^{-i\xi}}{i\xi} \right)^q, \quad \operatorname{Sc} q > 1. \qquad (*)$$

Setting $\Xi(\xi) := \dfrac{1 - e^{-i\xi}}{i\xi}$, the precise meaning of $(*)$ is

$$\begin{aligned} \widehat{B_q}(\xi) &= \Xi(\xi)^{\operatorname{Sc} q} \left(\cos(|v| \log \Xi(\xi)) + \frac{v}{|v|} \sin(|v| \log \Xi(\xi)) \right) \\ &= \widehat{B}_{\operatorname{Sc} q} \left(\cos(|v| \log \Xi(\xi)) + \frac{v}{|v|} \sin(|v| \log \Xi(\xi)) \right). \end{aligned}$$

(J. Hogan, P.M.: Quaternionic B-splines, J. Approx. Th. 224 (2017), 43–65)

Some Properties of Quaternionic B-Splines

Let B_q be a quaternionic B-splines with $\operatorname{Sc} q > \frac{1}{2}$.

- Decay: $B_q \in \mathcal{O}(|x|^{-\operatorname{Sc} q})$ as $|x| \to \infty$.
- Smoothness: $B_q \in H^{s,p}(\mathbb{R}, \mathbb{H}_\mathbb{C})$ for $1 \leq p \leq \infty$ and $0 \leq s < \operatorname{Sc} q + \frac{1}{p}$.
- Polynomial Reproduction: B_q reproduces polynomials up to order $\lceil \operatorname{Sc} q \rceil$.

Here the Bessel potential space $H^{s,p}(\mathbb{R}, \mathbb{H}_\mathbb{C})$ is defined by:

$$H^{s,p}(\mathbb{R}, \mathbb{H}_\mathbb{C}) = \left\{ f \in L^p(\mathbb{R}, \mathbb{H}_\mathbb{C}) : \mathcal{F}^{-1}[(1 + |\cdot|^2)^{s/2} \mathcal{F} f] \in L^p(\mathbb{R}, \mathbb{H}_\mathbb{C}) \right\}.$$

(J. Hogan, P.M.: Quaternionic B-splines, J. Approx. Th. 224 (2017), 43–65)

Time Domain Representation

Theorem (Hogan, M. 2017)

The quaternionic B-spline B_q $(q \in \mathbb{H}_{\mathbb{R}}, \operatorname{Sc} q > 1)$, *has the time domain representation*

$$B_q(x) = \frac{1}{\Gamma(q)} \sum_{k=0}^{\infty} (-1)^k \binom{q}{k} (x-k)_+^{q-1}, \quad x \in \mathbb{R}.$$

This equality holds in the sense of distributions and in $L^2(\mathbb{R}, \mathbb{H}_{\mathbb{R}})$.

(J. Hogan, P.M.: Quaternionic B-splines, J. Approx. Th. 224 (2017), 43–65)

An Integral Operator of Quaternionic Order

Let $\mathscr{D}$ be the classical first-order differentiation operator and $q = a + v \in \mathbb{H}_{\mathbb{R}}$ with $a > 0$.

Define $\mathscr{D}^{-q}$ by

$$\mathscr{D}^{-q} := \frac{1}{\Gamma(q)} \int_0^1 t^q e^{-t\mathscr{D}} \frac{dt}{t}.$$

where the operator $e^{-t\mathscr{D}}$ is defined on distributions:

Let $\varphi \in \mathscr{D}$ be a test function and $f \in \mathscr{D}'$.

Denote by $\langle \cdot, \cdot \rangle$ the pairing between distributions and test functions.

$$\langle \mathcal{F}(e^{-t\mathscr{D}} f), \varphi \rangle = \langle \mathcal{F}(f(\cdot - t), \varphi \rangle.$$

Thus, $e^{-t\mathscr{D}} f = T_t f = f(\cdot - t)$.

Differential Operators of Quaternionic Order

Define the quaternionic power $\mathscr{D}^q$ (with $q = a + v$ and $a > 0$) of the operator $\mathscr{D}$ on distributions $f \in \mathcal{D}'$ by

$$\mathcal{F}\mathscr{D}^q f := (-i\xi)^q \mathcal{F} f.$$

For $\varphi \in \mathcal{D}$ and $f \in \mathcal{D}'$,

$$\langle \mathscr{D}^{-q} \mathscr{D}^q f, \varphi \rangle = \langle f, \varphi \rangle.$$

A Differential Equation for B_q

Theorem (Hogan, M. 2017)

Quaternionic B-spline satisfies the distributional fractional differential equation

$$\mathscr{D}^q B_q = \sum_{k=0}^{\infty} (-1)^k \binom{q}{k} \delta_k,$$

where δ_k denotes the Dirac distribution supported on $k \in \mathbb{Z}_0^+$.

(J. Hogan, P.M.: Quaternionic B-splines, J. Approx. Th. 224 (2017), 43–65)

Splines of Quaternionic Orders

A spline of quaternionic order q with $\operatorname{Sc} q > 1$ is any solution of the fractional differential equation

$$\mathscr{D}^q B_q = \sum_{k=0}^{\infty} c_k \, \delta_k,$$

where δ_k denotes the Dirac delta distribution supported on $k \in \mathbb{Z}_0^+$ and $\{c_\ell : \ell \in \mathbb{N}\}$ is some bounded sequence in $\mathbb{H}_\mathbb{R}$.

And beyond...

- Hypercomplex B-splines[1]: $q = a + \sum_{i=1}^{n} v_i e_i$
- Clifford B-splines: $\left(\frac{1-e^{-\xi}}{\xi}\right)^n$, $\xi \in \mathbb{R}_n$, $n \in \mathbb{N}$
- $\vdots$

[1] P. M.: Splines and Fractional Differential Operators, to appear in Special Issue, IJWMIP

Thank you!