A family of integral operators on spaces of fractional order

P. J. Miana and J. Oliva-Maza

Universidad de Zaragoza

FCM 2019
3rd Fractional Calculus Meeting
September 25 th-27 th, Zaragoza

joliva@unizar.es

Index

Table of Contents

Weyl fractional calculus

Let $\alpha > 0$. Then, the definition of *Weyl fractional integral* of order α, $W_+^{-\alpha}$, for a suitable function f on $\mathbb{R}^+$, is given by:

$$W_+^{-\alpha} f(t) \;=\; \frac{1}{\Gamma(\alpha)} \int_t^{\infty} (s-t)^{\alpha-1} f(s)\, ds, \qquad t \in \mathbb{R}^+.$$

Weyl fractional calculus

Let $\alpha > 0$. Then, the definition of *Weyl fractional integral* of order α, $W_+^{-\alpha}$, for a suitable function f on $\mathbb{R}^+$, is given by:

$$W_+^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^{\infty} (s-t)^{\alpha-1} f(s)\,ds, \qquad t \in \mathbb{R}^+.$$

Weyl fractional calculus

Let $\alpha > 0$. Then, the definition of *Weyl fractional integral* of order α, $W_+^{-\alpha}$, for a suitable function f on $\mathbb{R}^+$, is given by:

$$W_+^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^{\infty} (s-t)^{\alpha-1} f(s) ds, \qquad t \in \mathbb{R}^+.$$

Similarly, the *Weyl fractional derivative* of order $\beta > 0$, is given by

$$W_+^{\beta} f(t) := (-1)^n \frac{d^n}{dt^n} W_+^{-(n-\beta)} f(t), \quad n = [\beta] + 1, t > 0,$$

again, defined for a suitable function f on $\mathbb{R}^+$.

Weyl fractional calculus

Let $\alpha > 0$. Then, the definition of *Weyl fractional integral* of order α, $W_+^{-\alpha}$, for a suitable function f on $\mathbb{R}^+$, is given by:

$$W_+^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} f(s) ds, \qquad t \in \mathbb{R}^+.$$

Similarly, the *Weyl fractional derivative* of order $\beta > 0$, is given by

$$W_+^{\beta} f(t) := (-1)^n \frac{d^n}{dt^n} W_+^{-(n-\beta)} f(t), \quad n = [\beta] + 1, t > 0,$$

again, defined for a suitable function f on $\mathbb{R}^+$.

In particular, $W_+^n = (-1)^n \frac{d^n}{dt^n}$, and $W_+^{-n} = (-1)^n \underbrace{\int_\infty^t \cdots \int_\infty^t}_{n \text{ times}} (\cdot) dt$.

Schwartz class and Weyl fractional calculus

Let $\alpha \in \mathbb{R}$. To simplify notation, we denote by W_+^α the respective Weyl fractional integral or derivative, depending on sign(α) (and $W_+^0 := Id$).

Schwartz class and Weyl fractional calculus

Let $\alpha \in \mathbb{R}$. To simplify notation, we denote by W_+^α the respective Weyl fractional integral or derivative, depending on $\operatorname{sign}(\alpha)$ (and $W_+^0 := Id$).

Let $\mathcal{S}_+$ denote the Scwhartz class on the positive real line, that is:

$$\mathcal{S}_+ = \{f \in \mathcal{C}^\infty(\mathbb{R}^+) : \sup_{x \in \mathbb{R}^+} |x^n f^{(m)}(x)| < \infty, \ \forall n, m \in \mathbb{N}\}.$$

Schwartz class and Weyl fractional calculus

Let $\alpha \in \mathbb{R}$. To simplify notation, we denote by W_+^α the respective Weyl fractional integral or derivative, depending on sign(α) (and $W_+^0 := Id$).

Let $\mathcal{S}_+$ denote the Scwhartz class on the positive real line, that is:

$$\mathcal{S}_+ = \{f \in \mathcal{C}^\infty(\mathbb{R}^+) : \sup_{x \in \mathbb{R}^+} |x^n f^{(m)}(x)| < \infty, \, \forall n, m \in \mathbb{N}\}.$$

Then, for any $\alpha \in \mathbb{R}$, W_+^α is well defined on $\mathcal{S}_+$, and is actually a bijection from $\mathcal{S}_+$ onto itself. For any $\alpha, \beta > 0$, the Weyl fractional operators are well defined.

Schwartz class and Weyl fractional calculus

Let $\alpha \in \mathbb{R}$. To simplify notation, we denote by W_+^α the respective Weyl fractional integral or derivative, depending on sign(α) (and $W_+^0 := Id$).

Let $\mathcal{S}_+$ denote the Scwhartz class on the positive real line, that is:

$$\mathcal{S}_+ = \{f \in \mathcal{C}^\infty(\mathbb{R}^+) : \sup_{x\in\mathbb{R}^+} |x^n f^{(m)}(x)| < \infty, \; \forall n, m \in \mathbb{N}\}.$$

Then, for any $\alpha \in \mathbb{R}$, W_+^α is well defined on $\mathcal{S}_+$, and is actually a bijection from $\mathcal{S}_+$ onto itself. For any $\alpha, \beta > 0$, the Weyl fractional operators are well defined.

Moreover, the (integro-differentiation) group property holds:

$$W_+^\alpha W_+^\beta = W_+^{\alpha+\beta} \qquad \forall \alpha, \beta \in \mathbb{R}.$$

Sobolev-Lebesgue spaces on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ on $\mathbb{R}^+$

Hardy's inequality:

$$\|W_+^{-\alpha} f\|_p \leq B\left(\alpha, \frac{1}{p}\right) \|(\cdot)^\alpha f(\cdot)\|_p.$$

So $W_+^{-\alpha} : L^p(\mathbb{R}^+, t^{p\alpha}) \to L^p$ is a (injective) well defined continuous map.

Sobolev-Lebesgue spaces on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ on $\mathbb{R}^+$

Hardy's inequality:

$$\|W_+^{-\alpha} f\|_p \leq B\left(\alpha, \frac{1}{p}\right) \|(\cdot)^\alpha f(\cdot)\|_p.$$

So $W_+^{-\alpha} : L^p(\mathbb{R}^+, t^{p\alpha}) \to L^p$ is a (injective) well defined continuous map.

Galé and Miana (2006), Royo (2008) introduced the Banach spaces $\mathcal{T}_p^{(\alpha)}(t^\alpha), \alpha > 0, p \geq 1, (\mathcal{T}_p^0(t^0) := L^p(\mathbb{R}^+))$ as the image of $W_+^{-\alpha}$, that is,

$$\mathcal{T}_p^{(\alpha)}(t^\alpha) = W_+^{-\alpha}(L^p(\mathbb{R}^+, t^{p\alpha})).$$

Sobolev-Lebesgue spaces on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ on $\mathbb{R}^+$

Hardy's inequality:

$$\|W_+^{-\alpha} f\|_p \leq B\left(\alpha, \frac{1}{p}\right) \|(\cdot)^\alpha f(\cdot)\|_p.$$

So $W_+^{-\alpha} : L^p(\mathbb{R}^+, t^{p\alpha}) \to L^p$ is a (injective) well defined continuous map.

Galé and Miana (2006), Royo (2008) introduced the Banach spaces $\mathcal{T}_p^{(\alpha)}(t^\alpha), \alpha > 0, p \geq 1, (\mathcal{T}_p^0(t^0) := L^p(\mathbb{R}^+))$ as the image of $W_+^{-\alpha}$, that is,

$$\mathcal{T}_p^{(\alpha)}(t^\alpha) = W_+^{-\alpha}(L^p(\mathbb{R}^+, t^{p\alpha})).$$

Moreover, $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ is the completion of the Schwartz class $\mathcal{S}_+$ under the norm:

$$\|f\|_{\alpha,p} := \frac{1}{\Gamma(\alpha+1)} \left(\int_0^\infty |W_+^\alpha f(t)|^p t^{\alpha p} dt\right)^{\frac{1}{p}} = \frac{1}{\Gamma(\alpha+1)} \|t^\alpha W_+^\alpha f(t)\|_p.$$

Sobolev-Lebesgue spaces on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ on $\mathbb{R}^+$

Then, it is suitable to regard the $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces as Banach spaces being formed by 'fractional primitives' of functions, so we can take the derivative of order α of any $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$, which lies in the weighted-space $L^p(\mathbb{R}^+, t^{p\alpha})$.

Sobolev-Lebesgue spaces on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ on $\mathbb{R}^+$

Then, it is suitable to regard the $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces as Banach spaces being formed by 'fractional primitives' of functions, so we can take the derivative of order α of any $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$, which lies in the weighted-space $L^p(\mathbb{R}^+, t^{p\alpha})$.

It follows that $\mathcal{T}_p^{\beta}(t^\beta) \hookrightarrow \mathcal{T}_p^{(\alpha)}(t^\alpha) \hookrightarrow L^p(\mathbb{R}^+)$, $\beta \geq \alpha \geq 0$, being the inclusions continuous maps.

$\mathcal{T}_p^{(\alpha)}(t^\alpha)$ is a module for the algebra $T_1^\alpha(t^\alpha)$ under convolution $*$.

$\mathcal{T}_2^{(\alpha)}(t^\alpha)$ is a RKHS if and only if $\alpha > 1/2$.

Sobolev-Lebesgue spaces on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ on $\mathbb{R}^+$

Then, it is suitable to regard the $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces as Banach spaces being formed by 'fractional primitives' of functions, so we can take the derivative of order α of any $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$, which lies in the weighted-space $L^p(\mathbb{R}^+, t^{p\alpha})$.

It follows that $\mathcal{T}_p^{\beta}(t^\beta) \hookrightarrow \mathcal{T}_p^{(\alpha)}(t^\alpha) \hookrightarrow L^p(\mathbb{R}^+)$, $\beta \geq \alpha \geq 0$, being the inclusions continuous maps.

$\mathcal{T}_p^{(\alpha)}(t^\alpha)$ is a module for the algebra $T_1^\alpha(t^\alpha)$ under convolution $*$.

$\mathcal{T}_2^{(\alpha)}(t^\alpha)$ is a RKHS if and only if $\alpha > 1/2$.

Note that $\mathcal{T}_p^{(0)}(t^0) = L^p(\mathbb{R}^+)$.

A family of integral operators on $L^p(\mathbb{R}^+)$

The operator $\mathcal{S}_{\beta,\mu,\lambda}$ given by

$$\mathcal{S}_{\beta,\mu,\lambda}f(t) \quad := \quad t^{\mu\lambda-\beta}\int_0^\infty \frac{s^{\beta-1}}{(s^\lambda + t^\lambda)^\mu} f(s)ds,$$

A family of integral operators on $L^p(\mathbb{R}^+)$

The operator $\mathcal{S}_{\beta,\mu,\lambda}$ given by

$$\mathcal{S}_{\beta,\mu,\lambda} f(t) \quad := \quad t^{\mu\lambda-\beta} \int_0^\infty \frac{s^{\beta-1}}{(s^\lambda + t^\lambda)^\mu} f(s) ds,$$

One obtains as special cases

$$\mathcal{S}_{1,1,1} f(t) \quad = \quad \mathcal{S} f(t) = \int_0^\infty \frac{1}{s+t} f(s) ds = (\mathcal{L} \circ \mathcal{L}) f(t),$$

A family of integral operators on $L^p(\mathbb{R}^+)$

The operator $\mathcal{S}_{\beta,\mu,\lambda}$ given by

$$\mathcal{S}_{\beta,\mu,\lambda}f(t) \quad := \quad t^{\mu\lambda-\beta}\int_0^\infty \frac{s^{\beta-1}}{(s^\lambda+t^\lambda)^\mu}f(s)ds,$$

One obtains as special cases

$$\begin{aligned}\mathcal{S}_{1,1,1}f(t) &= \mathcal{S}f(t)=\int_0^\infty \frac{1}{s+t}f(s)ds=(\mathcal{L}\circ\mathcal{L})f(t),\\ \mathcal{S}_{2,1,2}f(t) &= \mathcal{P}f(t)=\int_0^\infty \frac{s}{s^2+t^2}f(s)ds,\end{aligned}$$

A family of integral operators on $L^p(\mathbb{R}^+)$

The operator $\mathcal{S}_{\beta,\mu,\lambda}$ given by

$$\mathcal{S}_{\beta,\mu,\lambda}f(t) \quad := \quad t^{\mu\lambda-\beta}\int_0^\infty \frac{s^{\beta-1}}{(s^\lambda + t^\lambda)^\mu} f(s)ds,$$

One obtains as special cases

$$\begin{aligned}
\mathcal{S}_{1,1,1}f(t) &= \mathcal{S}f(t) = \int_0^\infty \frac{1}{s+t}f(s)ds = (\mathcal{L}\circ\mathcal{L})f(t),\\
\mathcal{S}_{2,1,2}f(t) &= \mathcal{P}f(t) = \int_0^\infty \frac{s}{s^2+t^2}f(s)ds,\\
\mathcal{S}_{1,1/2,2}f(t) &= \int_0^\infty \frac{1}{\sqrt{s^2+t^2}}f(s)ds.
\end{aligned}$$

Some particular cases

Hardy, Littlewood & Pólya (1934):
$\mathcal{S}_{\beta,\mu,\lambda}$ is a bounded operator on $L^p(\mathbb{R}^+), 1 \leq p < \infty$ if and only if $0 < \beta - \frac{1}{p} < \mu\lambda$,

Some particular cases

Hardy, Littlewood & Pólya (1934):
$\mathcal{S}_{\beta,\mu,\lambda}$ is a bounded operator on $L^p(\mathbb{R}^+), 1 \leq p < \infty$ if and only if $0 < \beta - \frac{1}{p} < \mu\lambda$, and

$$\|\mathcal{S}_{\beta,\mu,\lambda}\| = \frac{1}{\lambda} B\left(\mu - \frac{1}{\lambda}\left(\beta - \frac{1}{p}\right), \frac{1}{\lambda}\left(\beta - \frac{1}{p}\right)\right),$$

where $B(\cdot,\cdot)$ is the (Euler) Beta function.

Previous results on the spectrum set of $\mathcal{S}_{\beta,\mu,\lambda}$

Let $\mathcal{S}_{\beta,\mu,\lambda}$ be a generalized Stieltjes operator on $L^2(\mathbb{R}^+)$. We have the following results regarding its spectrum set:

Previous results on the spectrum set of $\mathcal{S}_{\beta,\mu,\lambda}$

Let $\mathcal{S}_{\beta,\mu,\lambda}$ be a generalized Stieltjes operator on $L^2(\mathbb{R}^+)$. We have the following results regarding its spectrum set:

Carleman (1923):

$$\sigma(\mathcal{S}) = \sigma(\mathcal{S}_{1,1,1}) = [0, \pi].$$

Previous results on the spectrum set of $\mathcal{S}_{\beta,\mu,\lambda}$

Let $\mathcal{S}_{\beta,\mu,\lambda}$ be a generalized Stieltjes operator on $L^2(\mathbb{R}^+)$. We have the following results regarding its spectrum set:

Carleman (1923):

$$\sigma(\mathcal{S}) = \sigma(\mathcal{S}_{1,1,1}) = [0, \pi].$$

Fedele and Pushnitski (2017):

$$\sigma\left(\mathcal{S}_{\beta+1,1+2\beta,1}\right) = [0, \pi_\beta], \qquad \beta > -3/2,$$

where

$$\pi_\beta := B\left(\beta + \frac{1}{2}, \beta + \frac{1}{2}\right) = \frac{\Gamma\left(\beta + \frac{1}{2}\right)^2}{\Gamma\left(2\beta + 1\right)}.$$

Table of Contents

C_0-groups on a complex Banach space I

Definition (C_0-group of operators)

Let X be a complex Banach space and let $\mathbb{B}(X)$ be the algebra of all bounded linear operators on X. A family $T = \{T(t)\}_{t\in\mathbb{R}}$ in $\mathbb{B}(X)$ is called a C_0-group if the following properties are satisfied:

1. $T(0) = 1_X$, the identity operator on X.
2. $T(t+s) = T(t)T(s)$, for every $t, s \in \mathbb{R}$.
3. $\text{lím}_{t\to 0} \| T(t)x - x\| = 0$, for all $x \in X$.

C_0-groups on a complex Banach space I

Definition (C_0-group of operators)

Let X be a complex Banach space and let $\mathbb{B}(X)$ be the algebra of all bounded linear operators on X. A family $T = \{T(t)\}_{t\in\mathbb{R}}$ in $\mathbb{B}(X)$ is called a C_0-group if the following properties are satisfied:

1. $T(0) = 1_X$, the identity operator on X.
2. $T(t+s) = T(t)T(s)$, for every $t, s \in \mathbb{R}$.
3. $\lim_{t\to 0} \| T(t)x - x\| = 0$, for all $x \in X$.

If, moreover,

$$\sup_{t\in\mathbb{R}} \| T(t)\| < \infty,$$

then $T = \{T(t)\}_{t\in\mathbb{R}}$ is called a C_0-group of uniformly bounded operators.

C_0-groups on a complex Banach space II

Definition (Infinitesimal generator)

Let $\{T(t)\}_{t\in\mathbb{R}}$ be C_0-group of operators. Then, for a dense subset $\mathcal{D}(X)$, the following operator is well defined:

$$\Lambda x := \lim_{\varepsilon\to 0} \frac{T(\varepsilon)x - x}{\varepsilon}.$$

C_0-groups on a complex Banach space II

Definition (Infinitesimal generator)

Let $\{T(t)\}_{t\in\mathbb{R}}$ be C_0-group of operators. Then, for a dense subset $\mathcal{D}(X)$, the following operator is well defined:

$$\Lambda x := \lim_{\varepsilon\to 0} \frac{T(\varepsilon)x - x}{\varepsilon}.$$

Moreover, Λ is a closed densely defined operator on X.

Subordination

Given a C_0-group of uniformly bounded operators on a Banach space X, $\{T(t)\}_{t\in\mathbb{R}}$,the following bounded linear map $\theta : L^1(\mathbb{R}) \to \mathcal{B}(X)$ is constructed [see, for example, Engel and Nagel (2000)]:

$$\theta(g)x = \int_{-\infty}^{\infty} g(t)T(t)xdt, \qquad x \in X, g \in L^1(R).$$

Moreover, $\|\theta(g)\| \leq \|g\|_1$.

Subordination

Given a C_0-group of uniformly bounded operators on a Banach space X, $\{T(t)\}_{t\in\mathbb{R}}$,the following bounded linear map $\theta : L^1(\mathbb{R}) \to \mathcal{B}(X)$ is constructed [see, for example, Engel and Nagel (2000)]:

$$\theta(g)x = \int_{-\infty}^{\infty} g(t)T(t)xdt, \qquad x \in X, g \in L^1(R).$$

Moreover, $\|\theta(g)\| \leq \|g\|_1$.
Seferoĝlu (2003) gave also a (more general) spectral mapping theorem:

$$\sigma(\theta(g)) = \overline{\widehat{g}(\sigma(i\Lambda))},$$

where $\widehat{g}$ is the Fourier transform of g, and Λ is the infinitesimal generator of $T(t)$.

C_0-group of isometries $T_{t,p}$ on $L^p(\mathbb{R}^+)$

Cowen (1984):
For $1 \leq p < \infty$, let's consider the set of isometries $(T_{t,p})_{t\in\mathbb{R}}$ given by

$$T_{t,p}f(s) = e^{-\frac{t}{p}} f(e^{-t}s), \qquad t \in \mathbb{R},\ f \in L^p(\mathbb{R}^+),$$

which is a C_0-group of uniformly bounded operators on $L^p(\mathbb{R}^+)$.

C_0-group of isometries $T_{t,p}$ on $L^p(\mathbb{R}^+)$

Cowen (1984):
For $1 \le p < \infty$, let's consider the set of isometries $(T_{t,p})_{t\in\mathbb{R}}$ given by

$$T_{t,p}f(s) = e^{-\frac{t}{p}} f(e^{-t}s), \qquad t \in \mathbb{R},\ f \in L^p(\mathbb{R}^+),$$

which is a C_0-group of uniformly bounded operators on $L^p(\mathbb{R}^+)$.
Moreover, the infinitesimal generator Λ of the group of isometries $(T_{t,p})_{t\in\mathbb{R}}$ is given by

$$\Lambda f(s) = -sf'(s) - \frac{1}{p} f(s), \qquad D(\Lambda) = \mathcal{T}_p^{(1)}(t^1),$$

C_0-group of isometries $T_{t,p}$ on $L^p(\mathbb{R}^+)$

Cowen (1984):
For $1 \leq p < \infty$, let's consider the set of isometries $(T_{t,p})_{t\in\mathbb{R}}$ given by

$$T_{t,p}f(s) = e^{-\frac{t}{p}} f(e^{-t}s), \qquad t \in \mathbb{R},\ f \in L^p(\mathbb{R}^+),$$

which is a C_0-group of uniformly bounded operators on $L^p(\mathbb{R}^+)$. Moreover, the infinitesimal generator Λ of the group of isometries $(T_{t,p})_{t\in\mathbb{R}}$ is given by

$$\Lambda f(s) = -sf'(s) - \frac{1}{p} f(s), \qquad D(\Lambda) = \mathcal{T}_p^{(1)}(t^1),$$

and its spectrum set by $\sigma(\Lambda) = i\mathbb{R}$.

C_0-group of isometries $T_{t,p}$ on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$

Lizama, Miana, Ponce, Sánchez-Lajusticia (2014)
For $1 \leq p < \infty$, $\alpha \geq 0$, let's consider the set of isometries $(T_{t,p})_{t\in\mathbb{R}}$ given by

$$T_{t,p}f(s) := e^{-\frac{t}{p}} f(e^{-t}s), \qquad t \in \mathbb{R},\ f \in \mathcal{T}_p^{(\alpha)}(t^\alpha),$$

C_0-group of isometries $T_{t,p}$ on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$

Lizama, Miana, Ponce, Sánchez-Lajusticia (2014)
For $1 \leq p < \infty$, $\alpha \geq 0$, let's consider the set of isometries $(T_{t,p})_{t\in\mathbb{R}}$ given by

$$T_{t,p}f(s) := e^{-\frac{t}{p}} f(e^{-t}s), \qquad t \in \mathbb{R},\ f \in \mathcal{T}_p^{(\alpha)}(t^\alpha),$$

which is a C_0-group of uniformly bounded operators on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$. Moreover, the infinitesimal generator Λ of the group of isometries $(T_{t,p})_{t\in\mathbb{R}}$ is given by

$$\Lambda f(s) = -sf'(s) - \frac{1}{p} f(s), \qquad D(\Lambda) = \mathcal{T}_p^{(\alpha+1)}(t^{\alpha+1}),$$

and its spectrum set by $\sigma(\Lambda) = i\mathbb{R}$.

Subordination of $\mathcal{S}_{\beta,\mu,\lambda}$

Aleman, Perfekt, Siskakis and Vukotic (work in progress)
Subordination of $\mathcal{S} = \mathcal{S}_{1,1,1}$ as an operator on $H^p(\mathbb{U})$.

Subordination of $\mathcal{S}_{\beta,\mu,\lambda}$

Aleman, Perfekt, Siskakis and Vukotic (work in progress)
Subordination of $\mathcal{S} = \mathcal{S}_{1,1,1}$ as an operator on $H^p(\mathbb{U})$.
Then, applying the change of variable $s = te^{-r}$, one can subordinate the generalized Stieltjes operator $\mathcal{S}_{\beta,\mu,\lambda}$ in terms of the C_0-group of isometries $T_{t,p}$:

$$\begin{aligned} \mathcal{S}_{\beta,\mu,\lambda}f(t) &= t^{\mu\lambda-\beta}\int_0^\infty \frac{s^{\beta-1}}{(s^\lambda+t^\lambda)^\mu}f(s)ds \\ &= \int_{-\infty}^{+\infty} \varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}(r)T_{r,p}f(t)dr, \quad t\geq 0, f\in\mathcal{T}_p^{(\alpha)}(t^\alpha), \end{aligned}$$

Subordination of $\mathcal{S}_{\beta,\mu,\lambda}$

Aleman, Perfekt, Siskakis and Vukotic (work in progress)
Subordination of $\mathcal{S} = \mathcal{S}_{1,1,1}$ as an operator on $H^p(\mathbb{U})$.
Then, applying the change of variable $s = te^{-r}$, one can subordinate the generalized Stieltjes operator $\mathcal{S}_{\beta,\mu,\lambda}$ in terms of the C_0-group of isometries $T_{t,p}$:

$$\begin{aligned} \mathcal{S}_{\beta,\mu,\lambda} f(t) &= t^{\mu\lambda-\beta} \int_0^\infty \frac{s^{\beta-1}}{(s^\lambda + t^\lambda)^\mu} f(s) ds \\ &= \int_{-\infty}^{+\infty} \varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}(r) T_{r,p} f(t) dr, \quad t \geq 0, f \in \mathcal{T}_p^{(\alpha)}(t^\alpha), \end{aligned}$$

where $$\varphi_{\nu,\mu,\lambda}(r) = \frac{e^{\nu r}}{(1 + e^{\lambda r})^\mu}, \qquad r \in \mathbb{R}.$$

Subordination of $\mathcal{S}_{\beta,\mu,\lambda}$

Aleman, Perfekt, Siskakis and Vukotic (work in progress)
Subordination of $\mathcal{S} = \mathcal{S}_{1,1,1}$ as an operator on $H^p(\mathbb{U})$.
Then, applying the change of variable $s = te^{-r}$, one can subordinate the generalized Stieltjes operator $\mathcal{S}_{\beta,\mu,\lambda}$ in terms of the C_0-group of isometries $T_{t,p}$:

$$\begin{aligned}
\mathcal{S}_{\beta,\mu,\lambda} f(t) &= t^{\mu\lambda-\beta} \int_0^\infty \frac{s^{\beta-1}}{(s^\lambda + t^\lambda)^\mu} f(s) ds \\
&= \int_{-\infty}^{+\infty} \varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}(r) T_{r,p} f(t) dr, \quad t \geq 0, f \in \mathcal{T}_p^{(\alpha)}(t^\alpha),
\end{aligned}$$

where $$\varphi_{\nu,\mu,\lambda}(r) = \frac{e^{\nu r}}{(1+e^{\lambda r})^\mu}, \qquad r \in \mathbb{R}.$$

Moreover, $\varphi_{\nu,\mu,\lambda} \in L^1(\mathbb{R}) \iff 0 < \nu < \lambda\mu$, and in that case

$$\begin{aligned}
\|\varphi_{\nu,\mu,\lambda}\|_1 &= \frac{1}{\lambda} B\left(\frac{\nu}{\lambda}, \mu - \frac{\nu}{\lambda}\right), \\
\widehat{\varphi_{\nu,\mu,\lambda}}(\xi) &= \frac{1}{\lambda} B\left(\frac{\nu}{\lambda} - i\frac{\xi}{\lambda}, \mu - \frac{\nu}{\lambda} + i\frac{\xi}{\lambda}\right).
\end{aligned}$$

Table of Contents

Boundedness

Then we have, for $1 \leq p < \infty$:

$$\begin{aligned} \|\mathcal{S}_{\beta,\mu,\lambda} f\|_{p,\alpha} &\leq \int_{-\infty}^{+\infty} \varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}(r) \|T_{r,p} f\|_{\alpha,p} dr \\ &= \|\varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}\|_1 \|f\|_{\alpha,p}. \end{aligned}$$

Boundedness

Then we have, for $1 \leq p < \infty$:

$$\begin{aligned} \|\mathcal{S}_{\beta,\mu,\lambda} f\|_{p,\alpha} &\leq \int_{-\infty}^{+\infty} \varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}(r) \|T_{r,p} f\|_{\alpha,p} dr \\ &= \|\varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}\|_1 \|f\|_{\alpha,p}. \end{aligned}$$

Even more, one can show that this upper bound is optimal, so $\mathcal{S}_{\beta,\mu,\lambda} \in \mathcal{B}\left(\mathcal{T}_p^{(\alpha)}(t^\alpha)\right)$, and

$$\|\mathcal{S}_{\beta,\mu,\lambda}\| = \|\varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}\|_1 = \frac{1}{\lambda} B\left(\mu - \frac{1}{\lambda}\left(\beta - \frac{1}{p}\right), \frac{1}{\lambda}\left(\beta - \frac{1}{p}\right)\right),$$

if and only if $0 < \beta - \frac{1}{p} < \mu\lambda$.

Spectrum set I

Recall that we have that

$$\sigma(\mathcal{S}_{\beta,\mu,\lambda}) = \sigma(\theta(\varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda})) = \overline{\widehat{\varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}}(\sigma(i\Lambda))},$$

where Λ is the infinitesimal generator of $(T_{t,p})_{t\in\mathbb{R}}$.

Spectrum set I

Recall that we have that

$$\sigma(\mathcal{S}_{\beta,\mu,\lambda}) = \sigma(\theta(\varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda})) = \overline{\widehat{\varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}}(\sigma(i\Lambda))},$$

where Λ is the infinitesimal generator of $(T_{t,p})_{t\in\mathbb{R}}$.

And the infinitesimal generator Λ of the group of isometries $(T_{t,p})_{t\in\mathbb{R}}$ has spectrum set $\sigma(\Lambda) = i\mathbb{R}$.

Spectrum set I

Recall that we have that

$$\sigma(\mathcal{S}_{\beta,\mu,\lambda}) = \sigma(\theta(\varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda})) = \overline{\widehat{\varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}}(\sigma(i\Lambda))},$$

where Λ is the infinitesimal generator of $(T_{t,p})_{t\in\mathbb{R}}$.

And the infinitesimal generator Λ of the group of isometries $(T_{t,p})_{t\in\mathbb{R}}$ has spectrum set $\sigma(\Lambda) = i\mathbb{R}$.

Finally,

$$\widehat{\varphi_{\nu,\mu,\lambda}}(\xi) = \frac{1}{\lambda} B\left(\frac{\nu}{\lambda} - i\frac{\xi}{\lambda}, \mu - \frac{\nu}{\lambda} + i\frac{\xi}{\lambda}\right).$$

Spectrum set II

Combining all of them, we obtain that, for any $0 < \beta - \frac{1}{p} < \mu\lambda$

$$\sigma(\mathcal{S}_{\beta,\mu,\lambda}) = \overline{\varphi_{\widehat{\mu\lambda-\beta+\frac{1}{p}},\mu,\lambda}(-\mathbb{R})}$$

$$= \left\{ \frac{1}{\lambda} B\left(\frac{1}{\lambda}\left(\beta - \frac{1}{p}\right) + it, \mu - \frac{1}{\lambda}\left(\beta - \frac{1}{p}\right) - it \right) : t \in \mathbb{R} \right\} \cup \{0\}.$$

regarding $\mathcal{S}_{\beta,\mu,\lambda}$ as an operator on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$, $1 \le p < \infty$, $\alpha \ge 0$.
In particular, for $\mathcal{T}_p^{(0)}(t^0) = L^p(\mathbb{R}^+)$.

Spectrum set II

Combining all of them, we obtain that, for any $0 < \beta - \frac{1}{p} < \mu\lambda$

$$\sigma(\mathcal{S}_{\beta,\mu,\lambda}) = \overline{\widehat{\varphi_{\mu\lambda-\beta+\frac{1}{p},\mu,\lambda}}(-\mathbb{R})}$$

$$= \left\{\frac{1}{\lambda}B\left(\frac{1}{\lambda}\left(\beta - \frac{1}{p}\right) + it, \mu - \frac{1}{\lambda}\left(\beta - \frac{1}{p}\right) - it\right) : t \in \mathbb{R}\right\} \cup \{0\}.$$

regarding $\mathcal{S}_{\beta,\mu,\lambda}$ as an operator on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$, $1 \leq p < \infty$, $\alpha \geq 0$.
In particular, for $\mathcal{T}_p^{(0)}(t^0) = L^p(\mathbb{R}^+)$.

This recovers the spectrum set given by Fedele and Pushnitski for $\mathcal{S}_{1+\beta,1+2\beta,1}$, $\beta > -3/2$ on $L^2(\mathbb{R}^+)$.

Table of Contents

Semifinite Hilbert transform on $L^p(\mathbb{R}^+)$

Let $1 < p < \infty$, and $f \in L^p(\mathbb{R}^+)$. Then, the so-called semifinite Hilbert transform $\mathcal{H}_+ : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ is defined as:

$$\mathcal{H}_+ f(t) := \text{p.v.} \frac{1}{\pi} \int_0^\infty \frac{f(s)}{t-s} ds, \quad t > 0.$$

Semifinite Hilbert transform on $L^p(\mathbb{R}^+)$

Let $1 < p < \infty$, and $f \in L^p(\mathbb{R}^+)$. Then, the so-called semifinite Hilbert transform $\mathcal{H}_+ : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ is defined as:

$$\mathcal{H}_+ f(t) := \text{p.v.} \frac{1}{\pi} \int_0^\infty \frac{f(s)}{t-s} ds, \quad t > 0.$$

It is in fact a well-defined, bounded operator on $L^p(\mathbb{R}^+)$, with norm

$$\|\mathcal{H}_+\|_{L^p(\mathbb{R}^+)} = \cot \frac{\pi}{2p^*}$$

Semifinite Hilbert transform on $L^p(\mathbb{R}^+)$

Let $1 < p < \infty$, and $f \in L^p(\mathbb{R}^+)$. Then, the so-called semifinite Hilbert transform $\mathcal{H}_+ : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ is defined as:

$$\mathcal{H}_+ f(t) := \text{p.v.} \frac{1}{\pi} \int_0^\infty \frac{f(s)}{t-s} ds, \quad t > 0.$$

It is in fact a well-defined, bounded operator on $L^p(\mathbb{R}^+)$, with norm

$$\|\mathcal{H}_+\|_{L^p(\mathbb{R}^+)} = \cot \frac{\pi}{2p^*} \qquad = \|\mathcal{H}\|_{L^p(\mathbb{R})},$$

Semifinite Hilbert transform on $L^p(\mathbb{R}^+)$

Let $1 < p < \infty$, and $f \in L^p(\mathbb{R}^+)$. Then, the so-called semifinite Hilbert transform $\mathcal{H}_+ : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ is defined as:

$$\mathcal{H}_+ f(t) := \text{p.v.} \frac{1}{\pi} \int_0^\infty \frac{f(s)}{t-s} ds, \quad t > 0.$$

It is in fact a well-defined, bounded operator on $L^p(\mathbb{R}^+)$, with norm

$$\|\mathcal{H}_+\|_{L^p(\mathbb{R}^+)} = \cot \frac{\pi}{2p^*} \qquad = \|\mathcal{H}\|_{L^p(\mathbb{R})},$$

where $\qquad p^* := \text{máx}\{p, q\}, \ \frac{1}{p} + \frac{1}{q} = 1.$

A convolution theorem for the Stieltjes transform

Yakubovich (1990), Srivastava and Kim Tuan (1995):
Let $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, $p,q,r\in[1,\infty)$. Then, the following map $\otimes : L^p(\mathbb{R}^+)\times L^q(\mathbb{R}^+)\to L^r(\mathbb{R}^+)$ is well defined:

$$(f\otimes g)(t) \quad := \quad f(t)\,\text{p.v.}\int_0^\infty \frac{g(s)}{s-t}ds + g(t)\,\text{p.v.}\int_0^\infty \frac{f(s)}{s-t}ds$$

A convolution theorem for the Stieltjes transform

Yakubovich (1990), Srivastava and Kim Tuan (1995):
Let $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $p, q, r \in [1, \infty)$. Then, the following map $\otimes : L^p(\mathbb{R}^+) \times L^q(\mathbb{R}^+) \to L^r(\mathbb{R}^+)$ is well defined:

$$\begin{aligned} (f \otimes g)(t) &:= f(t)\,\text{p.v.} \int_0^\infty \frac{g(s)}{s-t} ds + g(t)\,\text{p.v.} \int_0^\infty \frac{f(s)}{s-t} ds \\ &= -\pi \left(f(t)\mathcal{H}_+ g(t) + g(t)\mathcal{H}_+ f(t) \right), \qquad t > 0. \end{aligned}$$

A convolution theorem for the Stieltjes transform

Yakubovich (1990), Srivastava and Kim Tuan (1995):
Let $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, $p,q,r\in[1,\infty)$. Then, the following map $\otimes : L^p(\mathbb{R}^+)\times L^q(\mathbb{R}^+)\to L^r(\mathbb{R}^+)$ is well defined:

$$\begin{aligned}(f\otimes g)(t) &:= f(t)\,\text{p.v.}\int_0^\infty \frac{g(s)}{s-t}ds + g(t)\,\text{p.v.}\int_0^\infty \frac{f(s)}{s-t}ds \\ &= -\pi\left(f(t)\mathcal{H}_+g(t)+g(t)\mathcal{H}_+f(t)\right), \qquad t>0.\end{aligned}$$

Moreover, if $r>1$, then the following identity holds:

$$\mathcal{S}(f\otimes g)=\mathcal{S}f\,\mathcal{S}g.$$

Extension to the family of operators $\mathcal{S}_{\beta,\mu,\lambda}$

Let $r \geq 1$ and $m, n \in \mathbb{N}^+$ s.t. $0 < n - \frac{1}{r} < m$. Then:

$$\mathcal{S}_{n,m,1}(f \otimes g) = \sum_{i=n}^{m} \binom{m}{i} \sum_{j=0}^{i-n} \mathcal{S}_{i-j,m,1} f \, \mathcal{S}_{n+j,m,1} g - \sum_{i=0}^{n-2} \binom{m}{i} \sum_{j=0}^{n-2-i} \mathcal{S}_{n-j-1,m,1} f \, \mathcal{S}_{i+j+1,m,1} g,$$

Extension to the family of operators $\mathcal{S}_{\beta,\mu,\lambda}$

Let $r \geq 1$ and $m, n \in \mathbb{N}^+$ s.t. $0 < n - \frac{1}{r} < m$. Then:

$$\begin{aligned} \mathcal{S}_{n,m,1}(f \otimes g) = & \sum_{i=n}^{m} \binom{m}{i} \sum_{j=0}^{i-n} \mathcal{S}_{i-j,m,1} f \, \mathcal{S}_{n+j,m,1} g \\ & - \sum_{i=0}^{n-2} \binom{m}{i} \sum_{j=0}^{n-2-i} \mathcal{S}_{n-j-1,m,1} f \, \mathcal{S}_{i+j+1,m,1} g, \end{aligned}$$

where $f \in L^p(\mathbb{R}^+)$, $g \in L^q(\mathbb{R}^+)$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $p, q, r \in [1, \infty)$.

Extension to the family of operators $\mathcal{S}_{\beta,\mu,\lambda}$

Let $r \geq 1$ and $m, n \in \mathbb{N}^+$ s.t. $0 < n - \frac{1}{r} < m$. Then:

$$\mathcal{S}_{n,m,1}(f \otimes g) = \sum_{i=n}^{m} \binom{m}{i} \sum_{j=0}^{i-n} \mathcal{S}_{i-j,m,1} f \, \mathcal{S}_{n+j,m,1} g - \sum_{i=0}^{n-2} \binom{m}{i} \sum_{j=0}^{n-2-i} \mathcal{S}_{n-j-1,m,1} f \, \mathcal{S}_{i+j+1,m,1} g,$$

where $f \in L^p(\mathbb{R}^+)$, $g \in L^q(\mathbb{R}^+)$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $p, q, r \in [1, \infty)$.

What about $\mathcal{S}_{\beta,\mu,\lambda}$, $0 < \beta - \frac{1}{p} < \mu\lambda$?

Extension to $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces

To check that the previous identity holds on the family of spaces $\mathcal{T}_p^{(\alpha)}(t^\alpha)$, we need to check that in fact

$$\begin{aligned} \otimes : \mathcal{T}_p^{(\alpha)}(t^\alpha) \times \mathcal{T}_q^{(\alpha)}(t^\alpha) &\rightarrow \mathcal{T}_r^{(\alpha)}(t^\alpha) \\ f \times g &\rightarrow f \otimes g := -\pi \left(f\, \mathcal{H}_+ g + g\, \mathcal{H}_+ f \right), \end{aligned}$$

is well defined.

Extension to $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces

To check that the previous identity holds on the family of spaces $\mathcal{T}_p^{(\alpha)}(t^\alpha)$, we need to check that in fact

$$\begin{array}{rcl} \otimes : \mathcal{T}_p^{(\alpha)}(t^\alpha) \times \mathcal{T}_q^{(\alpha)}(t^\alpha) & \to & \mathcal{T}_r^{(\alpha)}(t^\alpha) \\ f \times g & \to & f \otimes g := -\pi \left(f\, \mathcal{H}_+ g + g\, \mathcal{H}_+ f \right), \end{array}$$

is well defined.

For that,

1. Is $\mathcal{H}_+$ well defined on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces?
2. Does pointwise product go from $\mathcal{T}_p^{(\alpha)}(t^\alpha) \times \mathcal{T}_q^{(\alpha)}(t^\alpha)$ to $\mathcal{T}_r^{(\alpha)}(t^\alpha)$?

Extension to $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces

To check that the previous identity holds on the family of spaces $\mathcal{T}_p^{(\alpha)}(t^\alpha)$, we need to check that in fact

$$\begin{aligned} \otimes : \mathcal{T}_p^{(\alpha)}(t^\alpha) \times \mathcal{T}_q^{(\alpha)}(t^\alpha) &\to \mathcal{T}_r^{(\alpha)}(t^\alpha) \\ f \times g &\to f \otimes g := -\pi \left(f\, \mathcal{H}_+ g + g\, \mathcal{H}_+ f \right), \end{aligned}$$

is well defined.
For that,

1. Is $\mathcal{H}_+$ well defined on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces?
2. Does pointwise product go from $\mathcal{T}_p^{(\alpha)}(t^\alpha) \times \mathcal{T}_q^{(\alpha)}(t^\alpha)$ to $\mathcal{T}_r^{(\alpha)}(t^\alpha)$?

Using properties of derivative (Leibniz formula, Hilbert transform of a derivative...), it holds for spaces of integer order $\alpha \in \mathbb{N}$.

Some new results on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces

For any $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, we have that:

$$\|\mathcal{H}_+\|_{\mathcal{T}_p^{(\alpha)}(t^\alpha)} = \|\mathcal{H}_+\|_{L^p(\mathbb{R}^+)} = \cot \frac{\pi}{2p^*}, \quad p^* := \text{máx}\{p, q\}.$$

So the semifinite Hilbert transform behaves in $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ as in $L^p(\mathbb{R}^+)$.

Some new results on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces

For any $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, we have that:

$$\|\mathcal{H}_+\|_{\mathcal{T}_p^{(\alpha)}(t^\alpha)} = \|\mathcal{H}_+\|_{L^p(\mathbb{R}^+)} = \cot \frac{\pi}{2p^*}, \quad p^* := \text{máx}\{p, q\}.$$

So the semifinite Hilbert transform behaves in $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ as in $L^p(\mathbb{R}^+)$.

Hölder's inequality holds on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces, that is, $f \cdot g \in \mathcal{T}_1^{(\alpha)}(t^\alpha)$ for $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$, $g \in \mathcal{T}_q^{(\alpha)}(t^\alpha)$, and then

$$\|f \cdot g\|_{\alpha,1} \leq C_{\alpha,p}\|f\|_{\alpha,p}\|g\|_{\alpha,q}.$$

Some new results on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces

For any $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, we have that:

$$\|\mathcal{H}_+\|_{\mathcal{T}_p^{(\alpha)}(t^\alpha)} = \|\mathcal{H}_+\|_{L^p(\mathbb{R}^+)} = \cot \frac{\pi}{2p^*}, \quad p^* := \text{máx}\{p, q\}.$$

So the semifinite Hilbert transform behaves in $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ as in $L^p(\mathbb{R}^+)$.

Hölder's inequality holds on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ spaces, that is, $f \cdot g \in \mathcal{T}_1^{(\alpha)}(t^\alpha)$ for $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$, $g \in \mathcal{T}_q^{(\alpha)}(t^\alpha)$, and then

$$\|f \cdot g\|_{\alpha,1} \leq C_{\alpha,p}\|f\|_{\alpha,p}\|g\|_{\alpha,q}.$$

What about if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ (generalized Hölder's inequality)?

Partial result

Let $p, q \in (1, \infty)$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$, and $m, n \in \mathbb{N}^+$ s.t. $0 < n - 1 < m$. Then:

$$\begin{aligned} \mathcal{S}_{n,m,1}(f \otimes g) = & \sum_{i=n}^{m} \binom{m}{i} \sum_{j=0}^{i-n} \mathcal{S}_{i-j,m,1} f \, \mathcal{S}_{n+j,m,1} g \\ & - \sum_{i=0}^{n-2} \binom{m}{i} \sum_{j=0}^{n-2-i} \mathcal{S}_{n-j-1,m,1} f \, \mathcal{S}_{i+j+1,m,1} g, \end{aligned}$$

where $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ and $g \in \mathcal{T}_q^{(\alpha)}(t^\alpha)$.

Table of Contents

Miscellaneous and future work

1. One can define the spaces of fractional order on the real line $\mathbb{R}$, $\mathcal{T}_p^{(\alpha)}(|t|^\alpha)$, and extend the generalized Stieltjes operator $\mathcal{S}_{\beta,\mu,\lambda}$ to these spaces, and everything works quite the same.

Miscellaneous and future work

1. One can define the spaces of fractional order on the real line $\mathbb{R}$, $\mathcal{T}_p^{(\alpha)}(|t|^\alpha)$, and extend the generalized Stieltjes operator $\mathcal{S}_{\beta,\mu,\lambda}$ to these spaces, and everything works quite the same.
2. $\mathcal{S}_{\beta,\mu,\lambda}$ on Hardy spaces on the half plane of integer order, $H_p^{(n)}$?

The end

Thank you
for your attention!

Spectral mapping theorem

Seferoĝlu (2003)
Let ω be a non-quasianalytic weight on $\mathbb{R}$ and let $T = (T(t))_{t\in\mathbb{R}}$ be a C_0-group on a Banach space X with generator Λ, such that $\|T(t)\| \leq \omega(t)$ for any $t \in \mathbb{R}$. Then

$$\sigma\left(\widehat{\phi}(T)\right) = \overline{\widehat{\phi}\left(\sigma(i\Lambda)\right)}, \qquad \phi \in regM_\omega(\mathbb{R})$$

where $\widehat{\phi}(T) = \int_{\mathbb{R}} T(t)d\phi(t)$

Note that, under this notation, we can write

$$\mathcal{S}_{\beta,\mu} = \widehat{\phi_{\mu-\beta+\frac{1}{p},\mu}}(T_{t,p})$$

Some identities

1. For $0 < \beta < \mu, 0 < \gamma < \nu, t \in \mathbb{R}$:

$$\frac{\phi_{\beta,\mu} * \phi_{\gamma,\nu}(t)}{B(\mu - \beta + \gamma, \nu - \gamma + \beta)} = e^{-(\mu-\beta)t}{}_2F_1(\mu, \mu-\beta+\gamma; \mu+\nu; 1-e^{-t})$$

2. For $\beta, \gamma, \mu, \nu > 0, t \in \mathbb{R}$ we have that

$$\phi_{\beta,\mu}*\psi_{\gamma,\nu}(t) = \gamma B(\gamma, \beta+\nu)\phi_{\beta,\mu}(t){}_2F_1\left(\mu, \gamma; \beta + \nu + \gamma; \frac{e^t}{e^t + 1}\right)$$

3. For $0 < \beta < 1 and t \in \mathbb{R}$:

$$\begin{aligned} \phi_{\beta,\mu} * \psi_{1,1-\beta}(t) &= \frac{e^{-(1-\beta)t}}{\mu - 1}\left(1 - \frac{1}{(1 + e^t)^{\mu-1}}\right), \\ \phi_{\beta,1} * \psi_{1,1-\beta}(t) &= e^{-(1-\beta)t}\log(1 + e^t). \end{aligned}$$

Gaussian hypergeometric function

$$ {}_2F_1(a,b;c;z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}, \qquad |z| < 1, $$

For $\Re c > \Re b > 0$, it can be analytically extended via the integral

$$ {}_2F_1(a,b;c;z) := \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 s^{b-1}(1-s)^{c-b-1}(1-zs)^{-a} ds, \qquad z \in \mathbb{C} \setminus [1,+\infty) $$