Symmetry, Functional Analisis and Non-linear Physics

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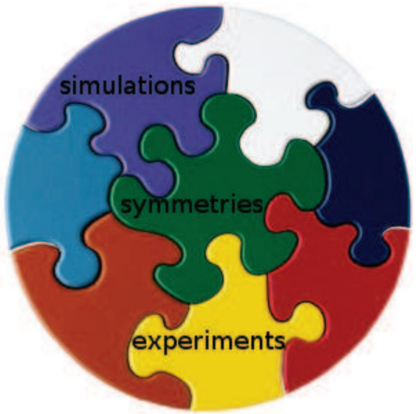
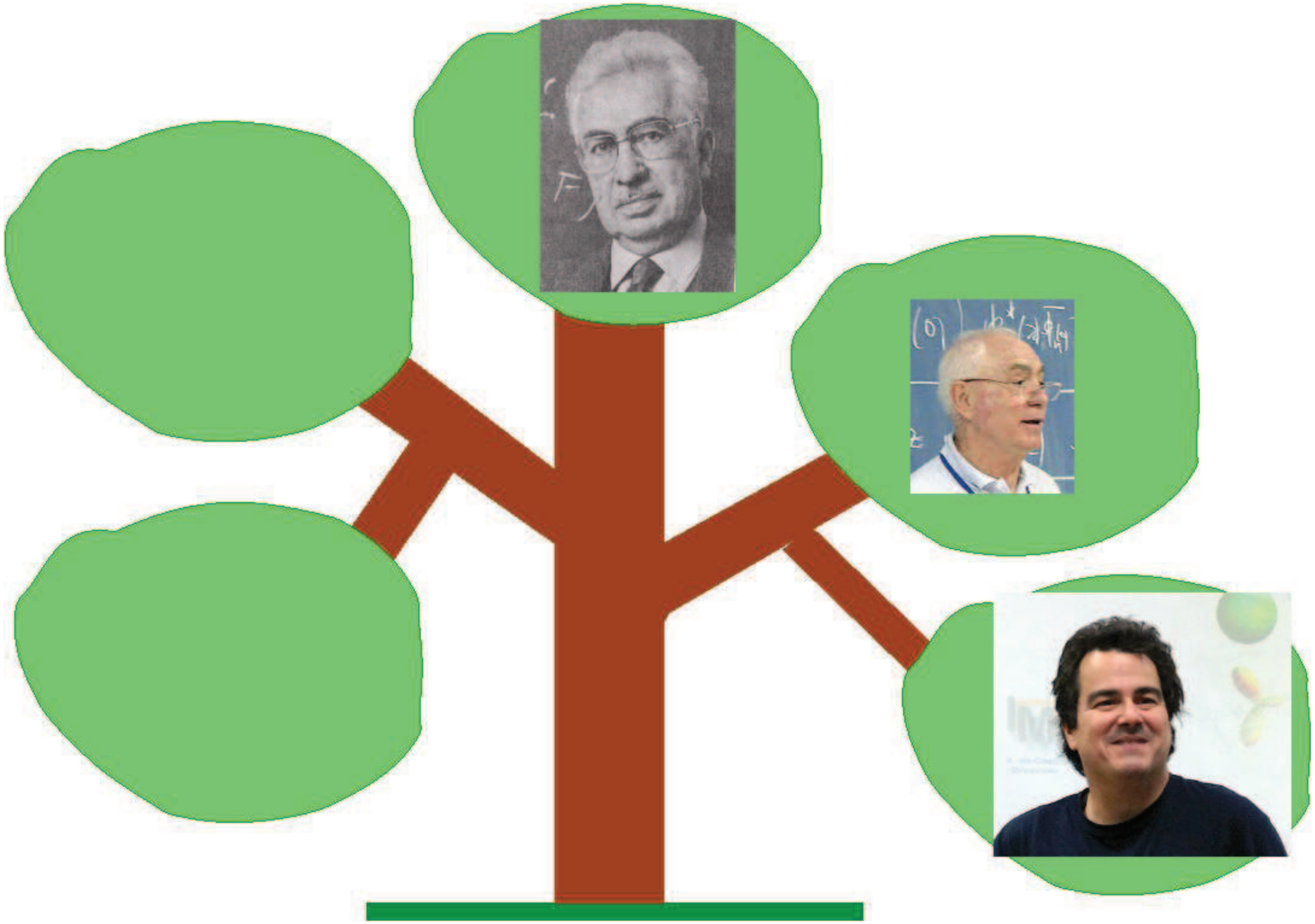


$$v \sim \epsilon_1^2 \epsilon_2 \cos(\phi + \theta)$$

E.I.T.A. 2014, Alquézar, 17-19 de octubre de 2014



Centenario del nacimiento de Luis Vigil

El Teorema de Galileo

Galileo: La filosofía [natural] está escrita en ese grandioso libro que tenemos abierto ante los ojos, (quiero decir, el universo), pero no se puede entender si antes no se aprende a entender la lengua, a conocer los caracteres en los que está escrito. Está escrito en lengua matemática y sus caracteres son triángulos, círculos y otras figuras geométricas, sin las cuales es imposible entender ni una palabra; sin ellos es como girar vanamente en un oscuro laberinto.

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⇒ **Corolario:** Si queremos conocer el mundo que nos rodea tenemos que saber Matemáticas.

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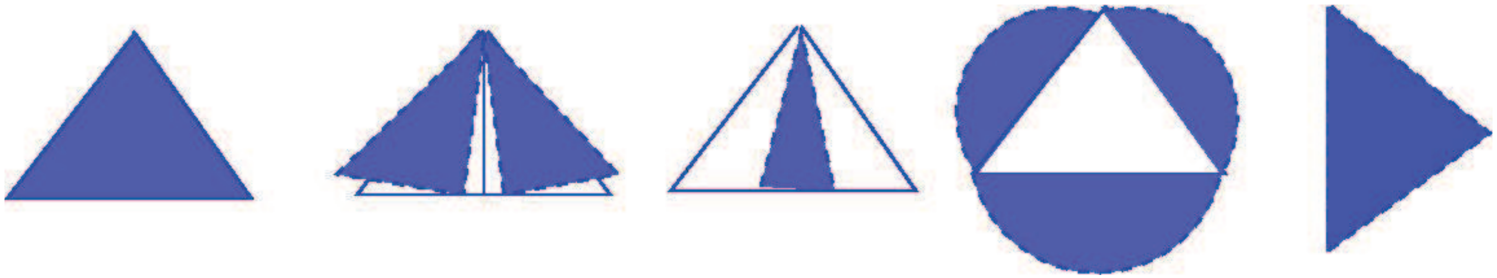
- **Invariance**: *a thing is symmetric if there is something we can do to it so that after we have done it, it looks the same as it did before.*

Let us consider a $\triangle$.

Transformations: We can do thing to our $\triangle$: $T : \triangle \mapsto \triangle'$: To do nothing, cut it, compress it, deform it, rotate it ...



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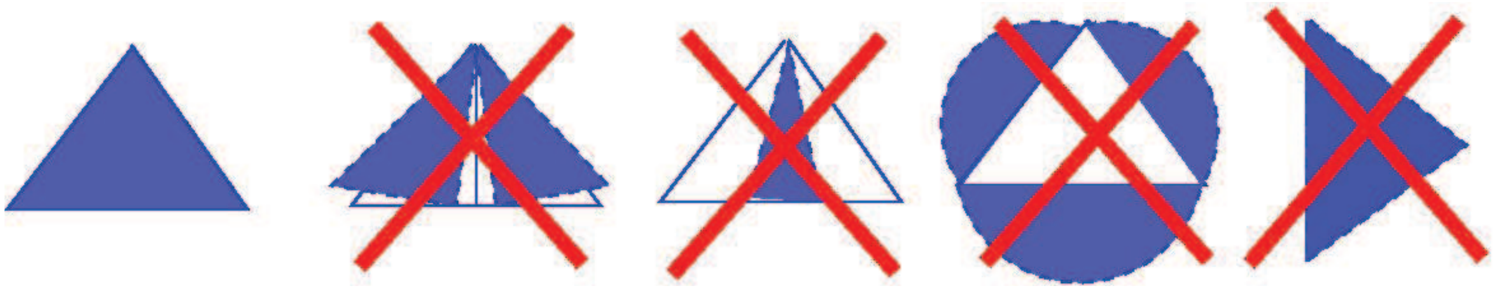

Invariance: The way the new $\triangle'$ looks should be identical to the original $\triangle$

Playing with the symmetry

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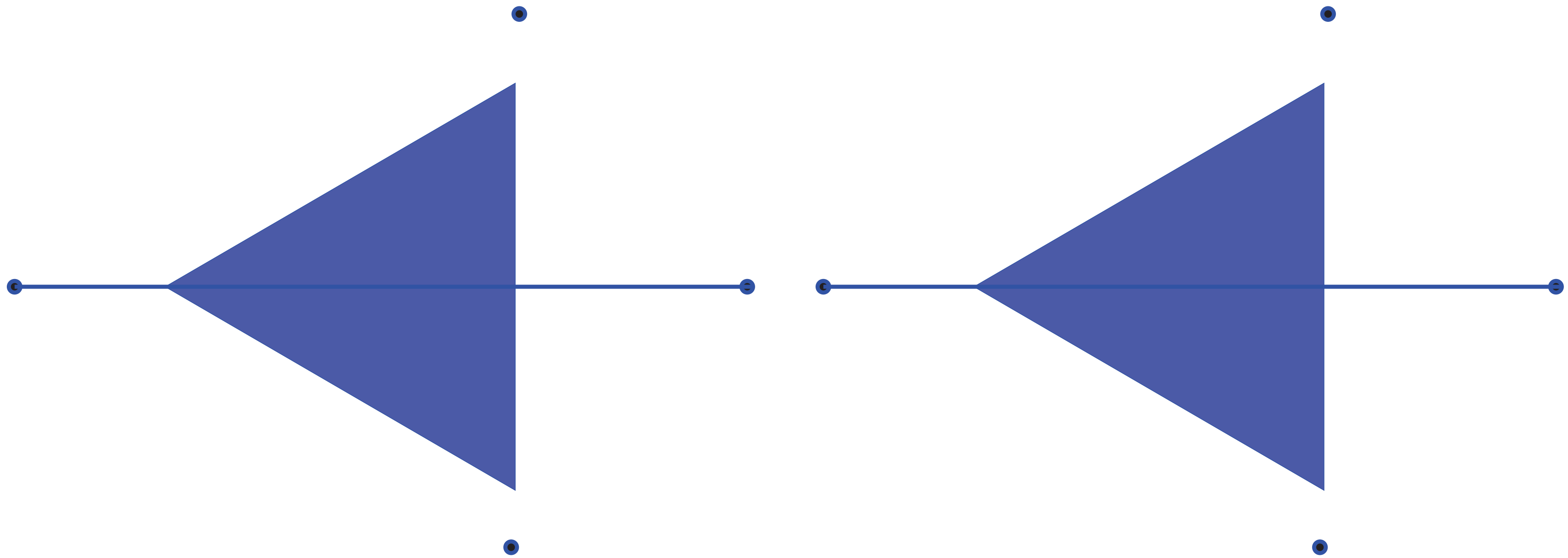

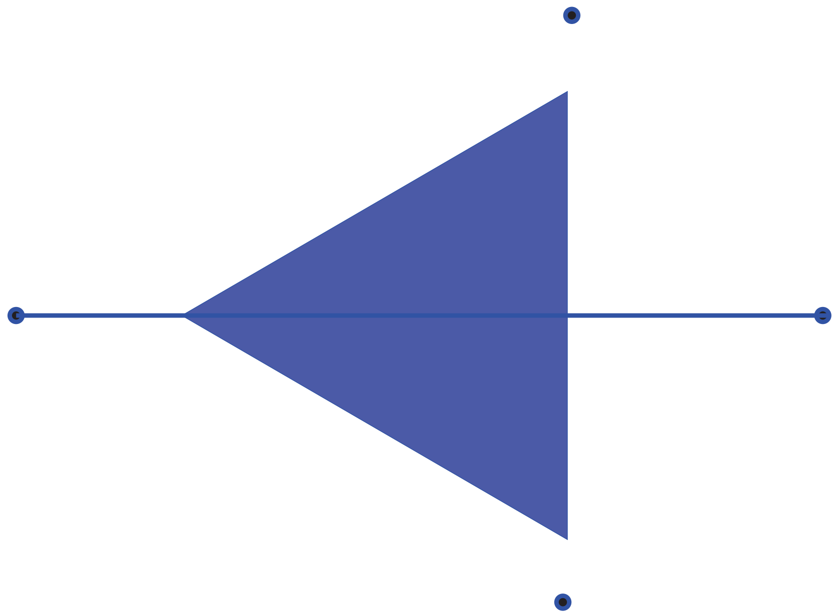

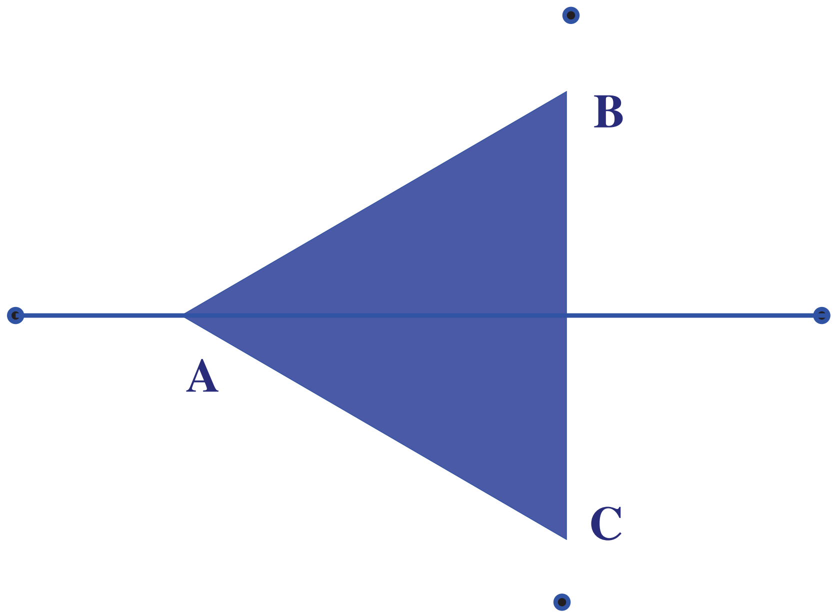
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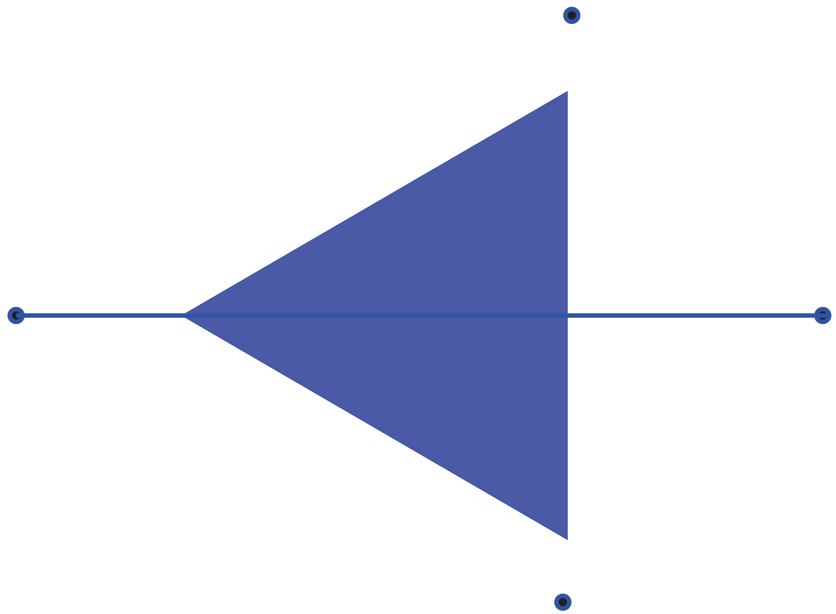

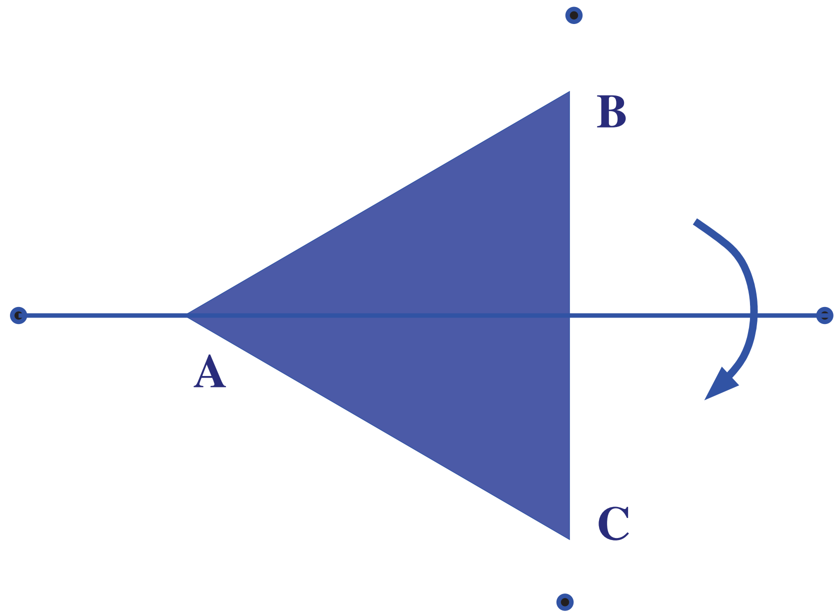
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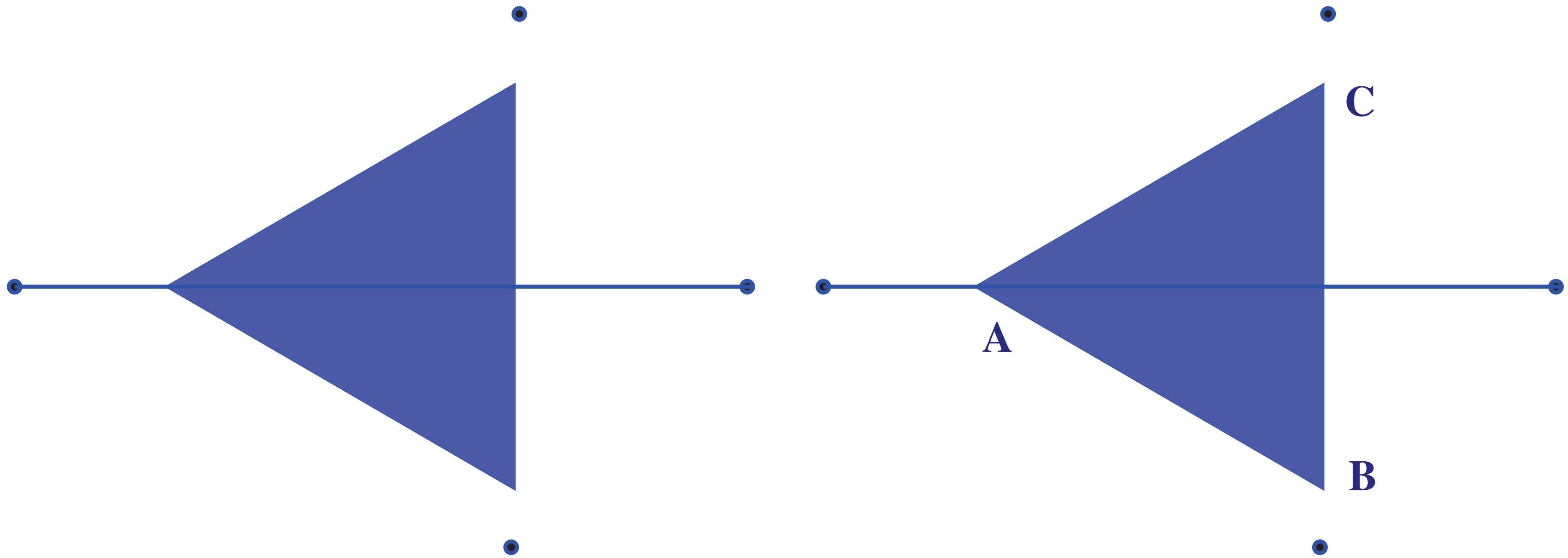
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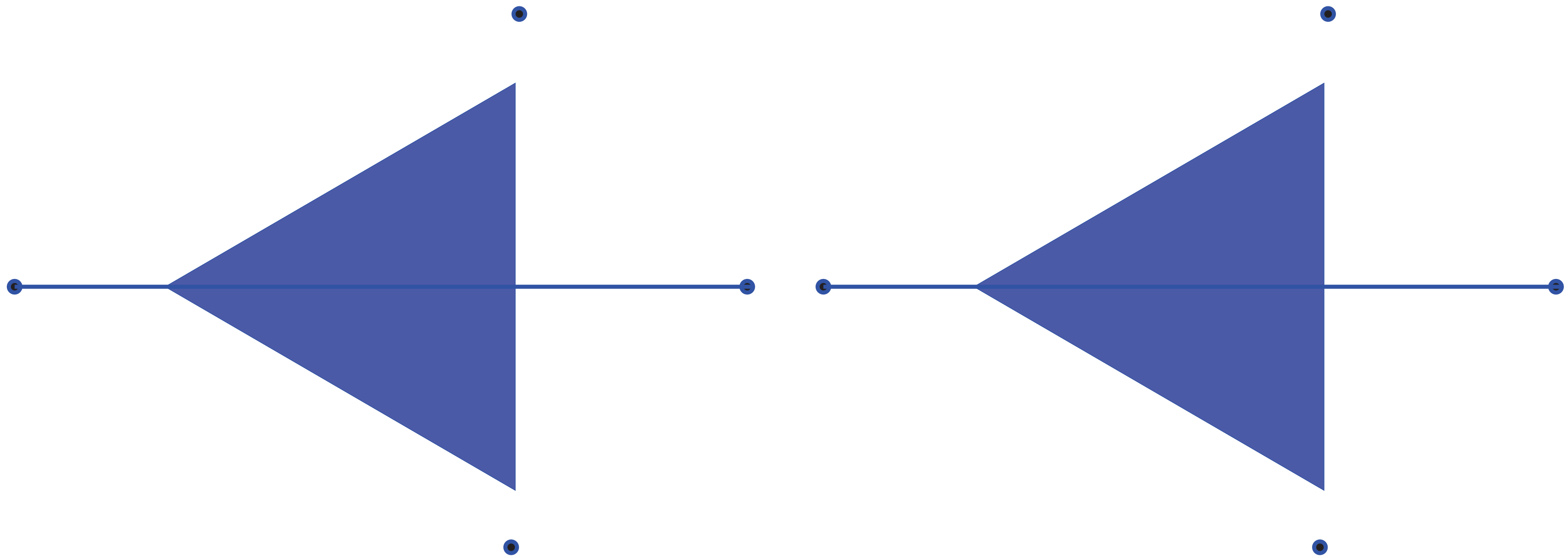

- What time-shift invariant of the measurement means?

Time-shift invariant: The experiments can be repeated at any time. Under the same conditions, the same results are obtained.

- System: device, equations, experiments, simulations...
- Input: harmonic (sinusoidal) functions
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- Input: harmonic (sinusoidal) functions
- Output: a number, which is an average measurement
- It has been found in experiments, simulations, . . . that the output has the same functional dependence on the parameters of the input

Why?

- The output is a functional of an arbitrary number, n, of external forces $f_i(t)$, $i = 1, 2, \ldots, n$
- In the following we will denote the output (functional) by $v[\mathbf{f}(t)]$, and the input will the the vector $\mathbf{f}(t)$.

An special emphasis will be done when $f_i(t) = \epsilon_i \cos(q_i \omega t + \phi_i)$, $i = 1, 2, \ldots, n$)

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- Implications in Physics: the experiments can be repeated at any time. Under the same conditions, the same results are obtained.
- Implications in Mathematics: the dependence of v of the amplitudes and phases of $\mathbf{f}$ are fixed and can be calculated as long as v is sufficiently *smooth*.

Ratchet effect: net motion of particles or solitons induced by periodic or random, zero-average external forces **f**.

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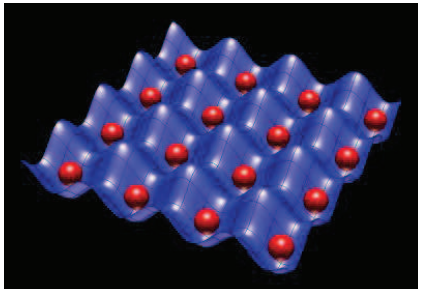

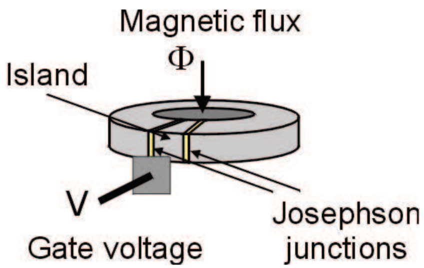

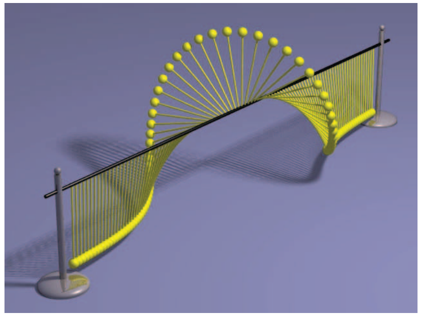

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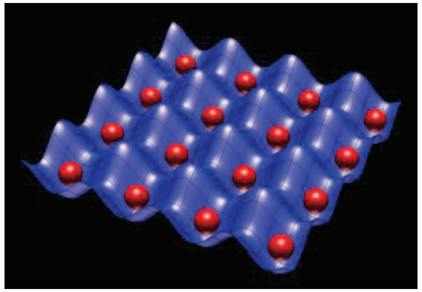

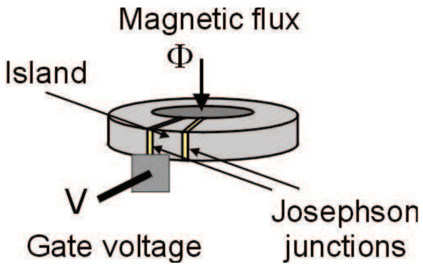

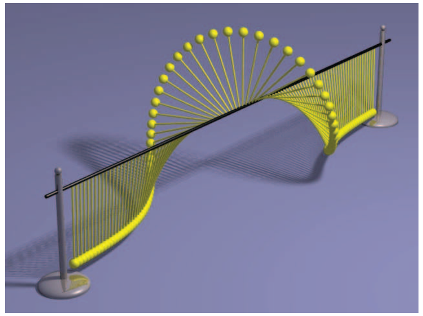

▶ Measurement of $x(t)$ (position of particles or non-linear waves)

Ratchet velocity $v \neq 0$ is an average velocity induced by $\langle \mathbf{f} \rangle = 0$

$$v := v[\mathbf{f}(t)] = \lim_{t\to+\infty} \frac{1}{t}\int_0^t \dot{x}(\tau)\,d\tau = \lim_{t\to+\infty} \frac{x(t)-x(0)}{t}.$$

Universality: “Same cause” $\Rightarrow$ “same effect”

A fact: Very different non-linear systems driven by the force $f(t) = \epsilon_1 \cos(p\Omega t) + \epsilon_2 \cos(q\Omega t + \phi)$

- Experiments: optical lattices, ferrofluids, semiconductors, Josephson junctions ...
- Simulations: dissipative dynamics of particles in symmetric potentials
- Simulations: dissipative dynamics of non-linear Schrödinger solitons in symmetric potentials
- Theory: dissipative dynamics of sine-Gordon and ϕ^4 solitons
- Theory: overdamped dynamics of particles in symmetric potentials

have the same behaviour:

$$v[\mathbf{f}(t)] = A\epsilon_1^p \epsilon_2^q \cos(p\phi + \theta_0)$$

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A special case: $f(t) = \epsilon_1 \cos(\Omega t) + \epsilon_2 \cos(2\Omega t + \phi)$

A first attempt for understanding this: The method of moments . . .

which is based on the assumption that $v[\mathbf{f}(t)]$ can be spanned as the Taylor series

$$\dot{x}(t) = \sum_{k=1}^{\infty} c_k (f(t))^{2k+1} \qquad \Rightarrow$$

$$v[\mathbf{f}(t)] = \langle \dot{x} \rangle = \sum_{k=1}^{\infty} c_{2k+1} \langle (f(t))^{2k+1} \rangle,$$

where here, we use the notation $\langle X \rangle := 1/T \int_0^T X(t)\, dt$.

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What happens with the phase! $\quad v[\mathbf{f}(t)] = A\epsilon_1^2 \epsilon_2 \cos(\phi + \boldsymbol{\theta_0})$

Why the methods of moments does not work?

Because the assumption that $v[\mathbf{f}(t)]$ can be spanned as Taylor series is wrong: $v[\mathbf{f}(t)]$ is a functional, thus we should use a functional series expansion: $\mathbf{f}(t) = (f_1(t), f_2(t))$

$$v[\mathbf{f}] = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \langle c_{n_1,n_2}(t_{11}, \cdots, t_{1n_1}, t_{21}, \ldots, t_{2n_2}) \times f_1(t_{11}) \cdots f_1(t_{1n_1}) f_2(t_{21}) \cdots f_2(t_{2n_2}) \rangle,$$

where the kernels c_{n_1,n_2} are real, T-periodic and symmetric. Here

$$\langle \Omega(t_1, \ldots, t_r) \rangle = \frac{1}{T^r} \int_0^T dt_1 \cdots \int_0^T dt_r \, \Omega(t_1, \ldots, t_r)$$

But the situation is worst ...

The method of moments **is wrong**. **Just one example:** Let be a system defined by (x is the position, u is the velocity)

$$\frac{d}{dt}\frac{Mu}{\sqrt{1-u^2}} = -f(t) - \gamma\frac{u}{\sqrt{1-u^2}}, \qquad \frac{dx}{dt} = u(t),$$

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donde $P(t) = \cdots$ and $v = B\epsilon_1^2\epsilon_2 \cos(2\phi_1 - \phi_2 + \underbrace{\boldsymbol{\theta_0}}_{\neq 0})$.

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If we use the force $f(t) = \begin{cases} \epsilon_1 & \text{if } 0 < t < T_l \\ 0 & \text{if } T_l < t < T - T_l \\ -\epsilon_1 & \text{if } T - T_l < t < T \end{cases} \Rightarrow$

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A little math ...

We recall that $v[\mathbf{f}]$ not an arbitrary functional. It is

a time-shift invariant functional, $\forall\tau$

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How the kernels are affected by this symmetry?

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$$\Downarrow$$

Fundamental property $\forall\tau\in(0,T)$:

$$c_{n_1,n_2}(t_{11}-\tau,\cdots,t_{1n_1}-\tau,t_{21}-\tau,\ldots,t_{2n_2}-\tau) = c_{n_1,n_2}(t_{11},\cdots,t_{1n_1},t_{21},\ldots,t_{2n_2})$$

- Theorem: Let $\mathbf{f}(t) = \{\epsilon_1 \cos(q_1\omega t + \phi_1), \epsilon_2 \cos(q_2\omega t + \phi_2)\}$.

If $\forall\tau\ v[\mathbf{f}(t+\tau)] = v[\mathbf{f}(t)]$ and $v[\mathbf{f}]$ *smooth* enough then

$$v[f_1, f_2] = C_0(\epsilon_1, \epsilon_2) + \sum_{\mathbf{x}\in\mathcal{D}_+} \epsilon_1^{|x_1|}\epsilon_2^{|x_2|} C_{\mathbf{x}}(\epsilon_1, \epsilon_2)\cos(\mathbf{x}\cdot\phi + \theta_{\mathbf{x}}(\epsilon_1, \epsilon_2)).$$

- $\mathbf{x} := (x_1, x_2) \in \mathcal{D}_+$: set of the solutions of the Diophantine Eq. $q_1x_1 + q_2x_2 = 0$ with $x_1 > 0$.
- $\phi = (\phi_1, \phi_2)$.
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Equivalently

$$v = C_0(\epsilon_1, \epsilon_2) + \sum_{k=1}^{\infty} (\epsilon_1^{q_2}\epsilon_2^{q_1})^k C_k(\epsilon_1, \epsilon_2)\cos(k(q_1\phi_2 - q_2\phi_1) + \theta_k(\epsilon_1, \epsilon_2))$$

- small amplitude limit: $v = C_0\epsilon_1^2\epsilon_2\cos(\phi+\theta_0)$, agrees with the simulations and experiments e.g. optical lattices, bi-harmonic force.

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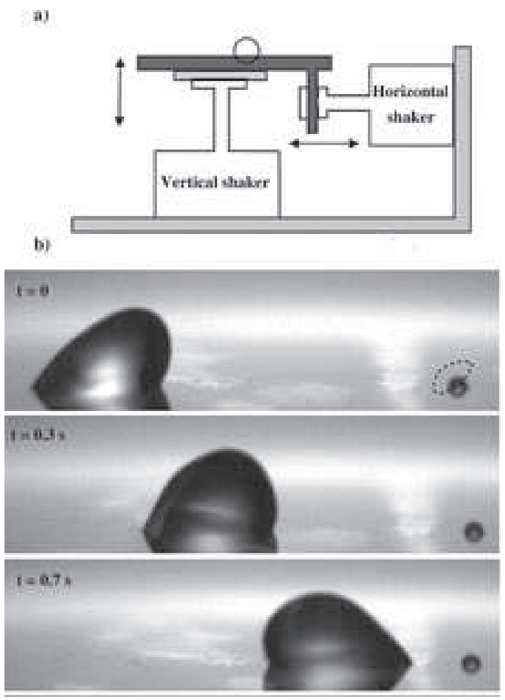
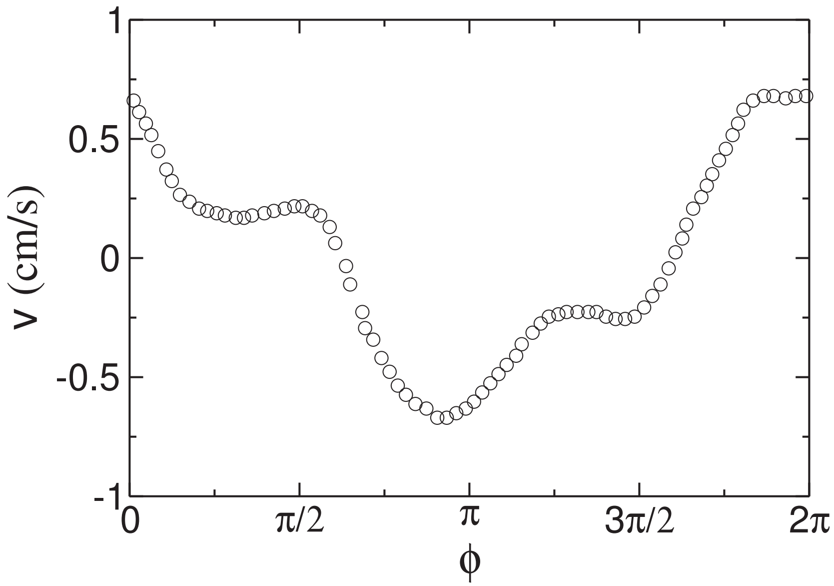
- large amplitudes: Prediction of the Theory: By increasing the amplitude of the forces, the direction of the motion can be changed! This has been observed in experiments: optical lattices, bi-harmonic force. Cubero, et. al., PRE 2010; also in simulations of the NLKG, etc.

$+$ a lot of interesting Math Corollaries ...

- (PRL, Noblin et. al), $f_v = \epsilon_1 \cos(\omega t + \phi)$, $f_h = \epsilon_2 \cos(\omega t)$.





$$v[-f_v, f_h] = -v[f_v, f_h]$$

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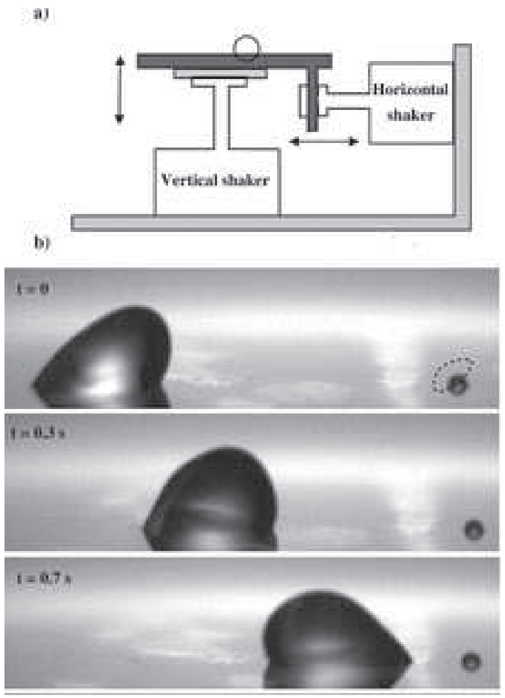

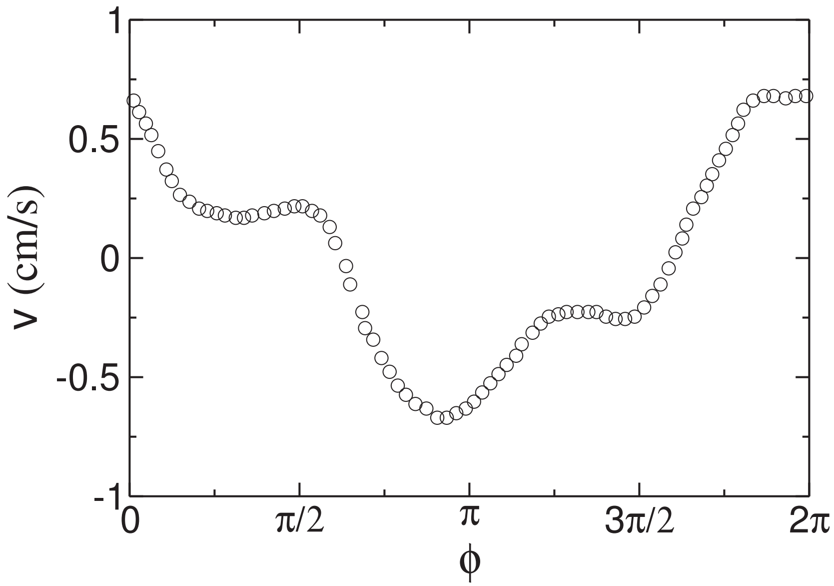

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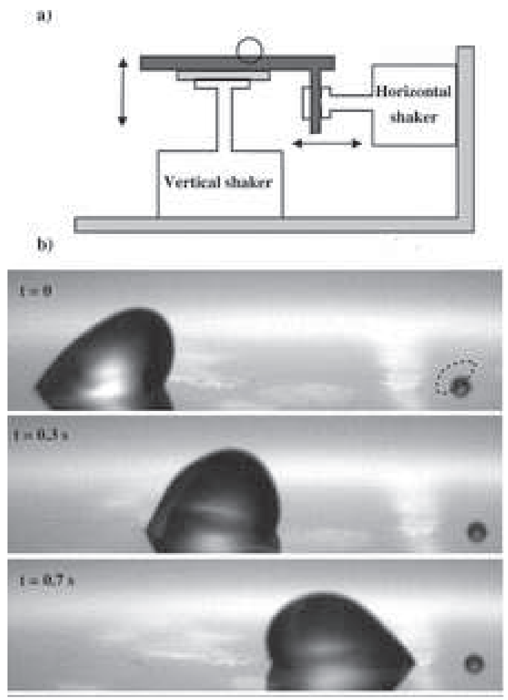

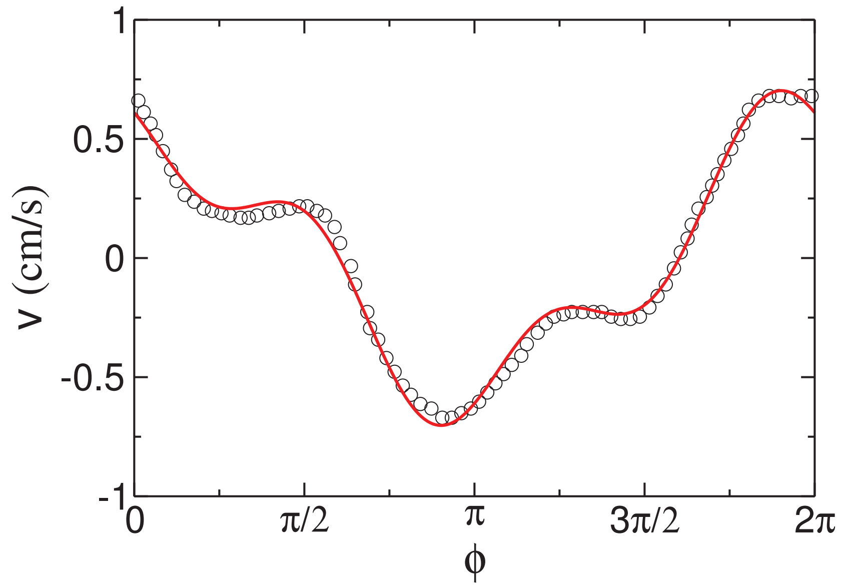

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- Theory vs experiment: $v = a\cos(\phi + b) + c\cos(3\phi + d)$

$$v = \sum_{k=1(odd)}^{\infty} (\epsilon_1^{q_2}\epsilon_2^{q_1})^k C_k(\epsilon_1,\epsilon_2)\cos(kq_1\phi + \theta_k(\epsilon_1,\epsilon_2)).$$

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- By setting $\phi = 0$, close to the Hamiltonian limit, $v = 0$. By increasing the damping, the transport is induced. We have been see this in the equation for a relativistic particle discussed before!

Main ingredients of the proof: $f_i = \epsilon_i \cos(q_i \omega t + \phi_i)$

$$v[\mathbf{f}] = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \langle c_{n_1,n_2}(t_{11}, \cdots, t_{1n_1}, t_{21}, \ldots, t_{2n_2}) \times f_1(t_{11}) \cdots f_1(t_{1n_1}) f_2(t_{21}) \cdots f_2(t_{2n_2}) \rangle,$$

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- The solutions of the Eq. $\mathbf{q} \cdot (\mathbf{k}-\mathbf{l}) = 0$ determine $\mathbf{k}, \mathbf{l}$
- For arbitrary n see (Cuesta, Quintero, Renato, PRX 2013)

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This set of periodic functions is invariant under these two transformations

$$\mathcal{T}_1 : \{t_0, \boldsymbol{\omega}, \boldsymbol{\varphi}, \boldsymbol{\epsilon}\} \longmapsto \{t_0, \boldsymbol{\omega}, \boldsymbol{\varphi} + \pi \mathbf{e}^{(j)}, \boldsymbol{\epsilon}^{(j)}\},$$
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where $\mathbf{e}^{(j)}$ canonical basis of $\mathbb{R}^n$ and the vector $\boldsymbol{\epsilon}^{(j)}$ is obtained from the vector $\boldsymbol{\epsilon}$ by replacing its jth component by $-\epsilon_j$.

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Let Υ represent a certain (physical) quantity of the system:

$$\Upsilon(t_0, \boldsymbol{\varphi}, \boldsymbol{\epsilon}) = \Upsilon(t_0, \boldsymbol{\varphi} + \pi \mathbf{e}^{(j)}, \boldsymbol{\epsilon}^{(j)}) = \Upsilon(0, \boldsymbol{\varphi} - \boldsymbol{\omega} t_0, \boldsymbol{\epsilon}),$$

By applying the first equality in the above equation twice, we see that Υ is periodic with respect to all the components of the vector $\boldsymbol{\varphi}$ with period 2π.

A simple proof ... Υ can be expanded in Fourier series as

$$\Upsilon(\zeta_0, \varphi, \epsilon) = \sum_{\boldsymbol{k} \in \mathbb{Z}^N} \upsilon_{\boldsymbol{k}}(\epsilon) e^{i(\varphi - \boldsymbol{\Omega}\zeta_0)\cdot \boldsymbol{k}} \quad \text{(in real form)}$$

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Case $t_0 = 0$: $f_j(t) = \epsilon_j \cos[\omega_j t + \varphi_j] = \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t)$ and analytic output: $\Upsilon = \sum_{\boldsymbol{l},\boldsymbol{r}\in\mathbb{N}_0^N} a_{\boldsymbol{l},\boldsymbol{r}} \prod_{j=1}^N \alpha_j^{l_j}\beta_j^{r_j}$, then computing the Fourier coeff.

$$\upsilon_{\boldsymbol{k}}(\boldsymbol{\epsilon}) = \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \frac{d^N\boldsymbol{\varphi}}{(2\pi)^N}\, \Upsilon(0, \boldsymbol{\varphi}, \boldsymbol{\epsilon}) e^{-i\boldsymbol{\varphi}\cdot\boldsymbol{k}}\,.$$

gives

Υ can be expanded as the following series

$$\Upsilon(\varphi,\epsilon) = \sum_{\mathbf{k}\in\mathcal{D}_+} |\gamma_{\boldsymbol{k}}(\epsilon)| \left(\prod_{j=1}^{N} \epsilon_j^{|k_j|}\right) \cos\left[\varphi\cdot\boldsymbol{k} + \chi_{\boldsymbol{k}}(\epsilon)\right] ,$$

$$\gamma_{\boldsymbol{k}}(\epsilon) = \sum_{\boldsymbol{p}\in\mathbb{N}_0^N} b_{\boldsymbol{k},\boldsymbol{p}} \prod_{j=1}^{N} \epsilon_j^{2p_j}$$

that is the same formula obtained using the functional analysis.

Thank you for your attention!

- Cuesta, Quintero, RAN, *Physical Review X* **3**, 041014 (2013).