GAUSS'S GAUSSIAN QUADRATURE

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METHODVS NOVA

INTEGRALIVM VALORES PER APPROXIMATIONEM INVENIENDI.

AVCTORE

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§1 to §6 (pages 3–11) review carefully the formulas by Cotes (1682–1716) (uniformly spaced nodes)

§7 to §12 (pages 11-21): construction of quadrature formulas with nonuniformly spaced nodes

- *Determinare* $\int y\,dx$ *inter limites datos* when several values of y are known. [No notation for functional dependence like modern $f(x)$.]

- *Integrale sumendum esse ab* $x = g$ *usque ad* $x = g + \Delta$.

- $t = \frac{x-g}{\Delta}$, $\Delta \int y\,dt$, *ab* $t = 0$ *usque ad* $t = 1$.

- $n + 1$ *valores dati* A, A', A'', A''',. . . , $A^{(n)}$.

- Corresponding values of t: a, a', a'', a''',. . . , $a^{(n)}$.

• Y *functionem algebraicam ordinis* n*:*

$$A\,\frac{(t-a')(t-a'')(t-a''')\cdots(t-a^{(n)})}{(a-a')(a-a'')(a-a''')\cdots(a-a^{(n)})}$$
$$+A'\,\frac{(t-a)(t-a'')(t-a''')\cdots(t-a^{(n)})}{(a'-a)(a'-a'')(a'-a''')\cdots(a'-a^{(n)})}$$
$$+\text{etc.}$$

such that if t is put equal to a, a', ..., Y takes the values A, A',... [Lagrange interpolating polynomial.]

• To compute $\int Y\,dt$ consider successively different parts of Y.

- Introduce

$$\begin{aligned} T &= (t-a)(t-a'')(t-a''')\cdots(t-a^{(n)}) \\ &= t^{n+1} + \alpha t^n + \alpha' t^{n-1} + \alpha'' t^{n-2} + \text{etc.} + \alpha^{(n)}. \end{aligned}$$

- then, the numerators in Y are $\frac{T}{t-a}, \frac{T}{t-a'}, \ldots$

- and the denominators M, M',... the values of $\frac{T}{t-a}, \frac{T}{t-a'}, \ldots$ at a, a', ... [Recall: no notation for functional dependence.] Thus:

$$Y = \frac{AT}{M(t-a)} + \frac{A'T}{M'(t-a')} + \text{etc}$$

• Let us compute M (similar for M', etc.)

$$T = t^{n+1} - a^{n+1} + \alpha(t^n - a^n) + \alpha'(t^{n-1} - a^{n-1}) + \text{etc.}$$

$$\begin{aligned}\frac{T}{t-a} = \; & t^n + at^{n-1} + aat^{n-2} + \text{etc.} + a^n \\ & + \alpha t^{n-1} + \alpha a t^{n-2} + \text{etc.} + \alpha a^{n-1} \\ & + \alpha' t^{n-2} + \text{etc.} + \alpha' a^{n-2} \\ & + \text{etc.etc.} \\ & + \alpha^{(n-1)}\end{aligned}$$

In $t = a$, this takes value $na^n + (n-1)\alpha a^{n-1} + \text{etc.} + \alpha^{(n-1)}$.

Thus M equals the value of $\frac{dT}{dt}$ at $t = a$, *uti etiam aliunde constat*.

- Next find *valorem integralis* $\int \frac{T\,dt}{t-a}$:

$$\begin{aligned}
\frac{1}{n+1} + \frac{a}{n} + \frac{aa}{n-1} + \text{etc.} + \qquad a^n \\
+\frac{\alpha}{n} + \frac{\alpha a}{n-1} + \text{etc.} + \alpha a^{n-1} \\
+\frac{\alpha'}{n-1} + \text{etc.} + \alpha' a^{n-2} \\
+\text{etc.etc.} \\
+\alpha^{(n-1)}.
\end{aligned}$$

[Which does not look too pretty?]

• *Quos terminos ordine sequente disponemus:*

$$a^n + \alpha a^{n-1} + \alpha' a^{n-2} + \text{etc.} + \alpha^{(n-1)}$$

$$+\text{etc.}$$

$$\frac{1}{n-1}(aa + \alpha a + \alpha')$$

$$\frac{1}{n}(a + \alpha)$$

$$\frac{1}{(n+1)},$$

and it is manifest that this is the result of multiplying T by $t^{-1} + \frac{1}{2}t^{-2} + +\frac{1}{3}t^{-3} + \frac{1}{4}t^{-4} + \text{etc.}$, discarding the terms with negative powers of t and replacing t by a. !!!

- Set

$$T\left(t^{-1} + \frac{1}{2}t^{-2} + +\frac{1}{3}t^{-3} + +\frac{1}{4}t^{-4} + \text{etc.}\right) = T' + T'',$$

where T' represents the [n-th degree] polynomial [in t] that the product contains. [Remember this formula. T' and T'' are crucial later. Note their coefficients are linear in the coefficients α, α', ..., of T.]

- Then $\int \frac{T\,dt}{t-a}$ equals the value of T' at $t = a$.

• To sum up, if R, R', ... denote the values of $\frac{T'}{\frac{dT}{dt}}$ at a, a', ...,
then $\int Y\,dt$ is

$$RA + R'A' + R''A'' + R'''A''' + \text{etc.} + R^{(n)}A^{(n)},$$

which multiplied by Δ will be the approximate value of $\int y\,dx$.

• Theory replicated, now using the variable $u = 2t - 1$ instead of t. Function $U = (u-b)(u-b')\ldots(u-b^{(n)})$ replaces T.

• Example: weights of Newton-Cotes formulas found with both t and u. The latter exploits symmetry $u \mapsto -u$.

• Next Gauss shows how to express the value of a rational function $\frac{Z}{\zeta}$ at the roots of a polynomial equation $\zeta' = 0$ as a polynomial in those roots. [Recall that the set (field) of rational expressions $\mathbb{Q}(\xi)$ coincides with the set of polynomials $\mathbb{Q}[\xi]$ when ξ is algebraic.]

• A fully detailed numerical example is given.

§13 to §14 (pages 22-24): error analysis

• For function t^m the error in the integral (from 0 to 1) is $k^{(m)}$ with

$$Ra^m + R'a'^m + \text{etc.} + R^{(n)}a^{(n)m} = \frac{1}{m+1} - k^{(m)}.$$

Multiply by t^{m-1} and sum to get:

$$\frac{R}{t-a} + \frac{R'}{t-a'} + \text{etc.} + \frac{R^{(n)}}{t-a^{(n)}} = t^{-1} + \frac{1}{2}t^{-2} + \frac{1}{3}t^{-3} + \text{etc.} - \theta,$$

with

$$\theta = kt^{-1} + k't^{-2} + k''t^{-3} + \text{etc.}$$

(k, k', *usque* $k^{(n)}$ *evanescere debere).*

[The sequences of true values $1/(m+1)$, approximate values $Ra^m + R'a'^m + \ldots$ and errors $k^{(m)}$ are represented here by their Z-transforms or generating functions. These are the Cauchy transforms $\int_{-\infty}^{\infty}(t-x)^{-1}d\mu(x)$ of the true measure dx in $[0,1]$, the measure $R\delta_a + R'\delta_{a'} + \cdots$ associated with the quadrature rule and the difference between both.]

[Note natural occurrence of the series $t^{-1} + \frac{1}{2}t^{-2} + \frac{1}{3}t^{-3} +$ `etc.`, which appeared above like deus ex machina.]

• Now recall $T(t^{-1} + (1/2)t^{-2} + \text{etc.}) = T' + T''$ to write

$$T\left(\frac{R}{t-a} + \frac{R'}{t-a'} + \text{etc.} + \frac{R^{(n)}}{t-a^{(n)}}\right) = T' + T'' - T\theta.$$

• *Pars prior ... est function integra ... ordinis* n whose values at a, a',. . . , are MR, $M'R'$, . . . , i.e. those of T'. So left-hand side is T'.

• Hence we obtain the important relation

$$T'' = T\theta.$$

Therefore the error coefficients may be computed from the expansion of T''/T.

• If $y = K + K't + K''tt + \text{etc.}$, the error in $\int y\,dt$ will be $k^{(n+1)}K^{(n+1)} + k^{(n+1)}K^{(n+1)} + \text{etc.}$ [Gauss can't write reminder of Taylor polynomial.]

§15 to §16 (pages 24–26): main idea

• For any values of a, a', . . . , the formula obtained is exact for orders $\leq n$.

• But for some values of a, a', . . . , the formula may be exact for higher degrees, as shown by the Cotes case with n even [something Gauss has discussed in detail in §6].

• For higher order we need to successively annihilate the error coefficients $k^{(n+1)}$, $k^{(n+2)}$, . . . (coefficients of t^{-n-1}, t^{-n-2}, . . . in θ). [i.e. it is a matter of $\theta = T''/T = (t^{-1} + \frac{1}{2}t^{-2} + \cdots) - T'/T$ being 'small'.]

• Equivalently, we need to successively annihilate the coefficients of t^{-1}, t^{-2}, ... in $T\theta$ i.e. in T''. [Recall these are linear in α, α', ..., hence the advantage in multiplying by T.]

• Since we have $n + 1$ free coefficients α, α', ..., we may annihilate the $n + 1$ leading coefficients of T'' and achieve degree $2n + 1$.

• In the simplest example, $n = 0$, *coefficiens unicus* of t^{-1} *in producto* $(t + \alpha)(t^{-1} + \frac{1}{2}t^{-2} + \frac{1}{3}t^{-3} + \text{etc.})$ *evanescere debet*. As this is $\frac{1}{2} + \alpha$, we have $\alpha = -\frac{1}{2}$ or $T = t - \frac{1}{2}$.

• The cases $n = 1$ and $n = 2$ (two and three linear equations to solve) also presented in detail; both in terms of t and u.

[Writing $T(t) \int_0^1 \frac{dx}{t-x} = \int_0^1 \frac{T(t)-T(x)}{t-x} dx + \int_0^1 \frac{T(x)dx}{t-x}$, we see that $T' = \int_0^1 \frac{T(t)-T(x)}{t-x} dx$, $T'' = \int_0^1 \frac{T(x)dx}{t-x}$. After expansion,

$$T'' = t^{-1} \int_0^1 T(x)dx + t^{-2} \int_0^1 xT(x)dx + \cdots$$

Thus annihilation of coefficients of T'' is equivalent to orthogonality of $T(x)$ to 1, x, ...]

[Note it is assumed without proof that the linear system for the coefficients has a unique solution. Also assumed that T found in this way has distinct real roots.]

[When the auxiliary variable u is used in lieu of t one has to approximate by U'/U

$$\varphi = u^{-1} + \frac{1}{3}u^{-3} + \frac{1}{5}u^{-5} + \text{etc.}$$

rather than $t^{-1} + \frac{1}{2}t^{-2} + \text{etc.}$ by T'/T]

- But this way, *qui calculos continuo molestiores adducit, hic ulterius non persequemur, sed ad fontem genuinum solutionis generalis progrediemur.*

§17 to §21 (pages 26–36): a better way

- *Proposita fractione continua*

$$\varphi = \cfrac{v}{w + \cfrac{v'}{w' + \cfrac{v''}{w'' + \text{etc.}}}}$$

formentur duae quantitatum series V, V', *etc.* W, W', *etc.*

$$V = 0 \qquad W = 1$$

$$V' = v \qquad W' = wW$$

$$V'' = w'V' + v'V \qquad W'' = w'W' + v'W$$

$$V''' = w''V'' + v''V' \qquad W''' = w''W'' + v''W'$$

etc.

- Then

$$
\begin{aligned}
\frac{V}{W} &= 0 \\
\frac{V'}{W'} &= \frac{v}{w} \\
\frac{V''}{W''} &= \frac{v}{w + \frac{v'}{w'}} \\
\frac{V'''}{W'''} &= \frac{v}{w + \frac{v'}{w' + \frac{v''}{w''}}}
\end{aligned}
$$

and so on.

- In addition, in the series

$$\frac{v}{WW'} - \frac{vv'}{W'W''} + \frac{vv'v''}{W''W'''} - \frac{vv'v''v'''}{W'''W^{iv}} + \text{etc.}$$

terminum primum $= \frac{V'}{W'}$

summam duorum terminum primorum $= \frac{V''}{W''}$

summam trium terminum primorum $= \frac{V'''}{W'''}$

and so on. Similarly we represent *differentia inter* φ and $\frac{V'}{W'}$, $\frac{V''}{W''}$, etc.

[Recall that in terms of the auxiliary variable u the aim is to approximate by a rational function U'/U (U of degree $n+1$, U' of degree n) the series

$$\varphi = u^{-1} + \frac{1}{3}u^{-3} + \frac{1}{5}u^{-5} + \text{etc.}]$$

- *E formula 33 Disquisitionum generalium circa seriem infinitam . . .* , [on the hypergeometric series (1812)] we transform φ into

$$\cfrac{1}{u - \cfrac{\frac{1}{3}}{u - \cfrac{\frac{2\cdot 2}{3\cdot 5}}{u - \cfrac{\frac{3\cdot 3}{5\cdot 7}}{u - \cfrac{\frac{4\cdot 4}{7\cdot 9}}{u - \text{etc.}}}}}}$$

- Here $v = 1$, $v' = -\frac{1}{3}$, $v'' = -\frac{4}{15}$, etc. and $w = w' = w''\text{etc.} = u$.

- So $W = 1$, $W' = u$, $W'' = uu - \frac{1}{3}$, $W''' = u^3 - \frac{3}{5}u$, etc. [These are the monic Legendre polynomials, generated from the three term recursion!]

- And $V = 0$, $V' = 1$, $V'' = u$, $V''' = uu - \frac{4}{15}$, etc. [The associated polynomials of the three term recursion!]

• If $\varphi - \frac{V^{(m)}}{W^{(m)}}$ *in seriem descendentem convertitur*, the first term is

$$\frac{2 \cdot 2 \cdot 3 \cdot 3 \cdots m \cdot m \, u^{-(2m+1)}}{3 \cdot 3 \cdots (2m-1)(2m-1)}.$$

[In modern terminology, $\frac{V^{(m)}}{W^{(m)}}$ is the Padé approximation to φ of degree $(m-1, m)$.] Thus if we set $U = W^{(n+1)}$ then $U\varphi$ is free of the powers $u^{-1}, \ldots, u^{-(n+1)}$.

• Therefore the abscissas have to be chosen as the roots of the equation $W^{(n+1)} = 0$. [Zeros of Legendre polynomial.]

Next Gauss:

• Provides a closed form expression for the monic Legendre polynomials and discusses the relation to the hypergeometric function.

• Presents similar analysis for t in lieu of u. [T is of course the Legendre polynomial shifted to $[0, 1]$.]

• Gives explicit expression for the polynomial that yields the weights.

[The relation

$$T' = \int_0^1 \frac{T(t) - T(x)}{t - x} dx$$

we found before (resp. the corresponding formula that expresses U' in terms of U) is the well-known formula that relates the associated (or numerator) polynomials to the shifted Legendre polynomials T (resp. Legendre polynomials U). I am thankful to F. Marcellán for this observation.]

§22 to §23 (pages 36–40): using the rules

- For $n = 0, \ldots, 6$ (one to seven nodes). Gauss provides:

1. Polynomials U, U', T, T'.

2. Abscissas a, a', . . . with 16 significant digits.

3. Weights R, R', . . . with 16 significant digits. (For $n \geq 3$ also decimal logarithm with 10 significant digits.)

4. The polynomial that gives the weights.

5. The leading coefficient of the expansion of the error.

• *Methodi nostrae efficaciam ab oculos ponemos computando valores integralis* $\int \frac{dx}{\log x}$ *ab* $x = 100000$ *usque ad* $x = 200000$ with rules with 1 to 7 nodes: (Bessel had computed 8406.24312)

8390.394608

8405.954599

8406.236775

8406.242970

8406.243117

8406.243121

8406.2431211

[There are 8392 prime numbers in the interval.]