Porous medium equations with nonlocal pressure

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joint work with

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1. The Porous Medium Equation

2. Nonlocal Diffusion
   - Nonlocal porous medium type equations

3. Our work
   - Existence of solutions
   - Positivity results
   - Other models
   - Transformations

The Porous Medium Equation

Derivation of the Porous Medium Equation

- Physical Model: a continuum (fluid or population) with density distribution $u(x,t) \geq 0$ and velocity field $\mathbf{v}(x,t)$.
- Continuity equation $u_t = \nabla(u \cdot \mathbf{v})$.
- Darcy's law (fluids in porous media): $\mathbf{v} = -\nabla p$.
- Leibenzon and Muskat (1930):

$$p = f(u),$$

  where f is a nondecreasing scalar function. $f(u)$ is linear when the flow is isothermal and is a higher power of u when the flow is adiabatic, i.e. $f(u) = cu^{m-1}$ with $c > 0$ and $m > 1$.
- $f(u) = cu \longrightarrow$ Boussinesq (1903) modelling water infiltration $\longrightarrow$ $u_t = c\Delta u^2$.
- The model $u_t = (c/m)\,\Delta u^m$.

The Porous Medium Equation $u_t = \Delta u^m$.

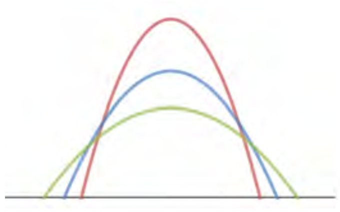
Porous Medium Equation / Fast Diffusion Equation

PME/FDE $u_t(x,t) = \Delta u^m(x,t) \quad x \in \mathbb{R}^N,\ t > 0$

Self Similar solutions: $\mathcal{U}(x,t) = t^{-\frac{N}{N(m-1)+2}} F\left(|x| t^{-\frac{1}{N(m-1)+2}}\right)$

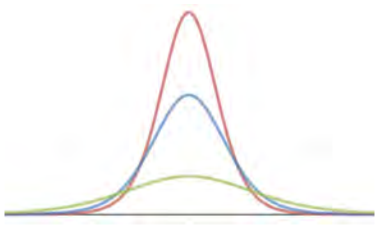
Slow Diffusion $m > 1$

$F(y) \sim (R^2 - |y|^2)_+^{1/(m-1)}$



Fast Diffusion $m < 1$

$F \sim (R^2 + |y|^2)^{-1/(1-m)}$



Nonlocal Diffusion

Definition of the Fractional Laplacian

Several equivalent definitions of the nonlocal operator $(-\Delta)^s$ (Laplacian of order $2s$):

1. Fourier transform $\widehat{(-\Delta)^s} g(\xi) = (2\pi|\xi|)^{2s}\hat{g}(\xi)$.

   [can be used for positive and negative values of s]

2. Singular Kernel $(-\Delta)^s g(x) = c_{N,s}$ P.V. $\displaystyle\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,dz$

   [can be used for $0 < s < 1$, where $c_{N,s}$ is a normalization constant.]

3. Heat semigroup

$$(-\Delta)^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta}g(x) - g(x)\right) \frac{dt}{t^{1+s}}\,.$$

4. Generator of the $2s$-stable Levy process:

$$(-\Delta)^s g(x) = \lim_{h\to 0} \frac{1}{h}\mathbb{E}[g(x) - g(x+X_h)].$$

One of the first nonlocal porous medium models

$$u_t + (u\,\mathcal{H}(u))_x = 0, \qquad x \in \mathbb{R},\ t > 0$$

$$\Downarrow \quad v = \int_{-\infty}^{x} u\,dx$$

$$v_t + |v_x|(-\Delta)^{1/2} v = 0, \qquad x \in \mathbb{R},\ t > 0$$

where $\mathcal{H}$ is the Hilbert transform:

$$\mathcal{H} = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{u(y)}{x-y} dy \Leftrightarrow \widehat{\mathcal{H}(u)}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{u}(\xi) \;\Rightarrow\; \partial_x \mathcal{H} = (-\Delta)^{1/2}.$$

P.Biler, G.Karch and R.Monneau *Nonlinear Diffusion of Dislocation Density and Self-Similar Solutions*, Commun. Math. Phys. 294, 145–168 (2010).

A.K. Head, *Dislocation group dynamics III. Similarity solutions of the continuum approximation*. Phil. Mag. 26, 65–72 (1972)

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$$u_t + (u\,\partial_x\,(-\Delta)^{-s}(u))_x = 0, \qquad x \in \mathbb{R},\ t > 0 \qquad \text{(P)}$$

$$\Downarrow \quad v = \int_{-\infty}^{x} u\,dx$$

$$v_t + |v_x|(-\Delta)^{1-s} v = 0, \qquad x \in \mathbb{R},\ t > 0 \qquad \text{(IP)}$$

where $\partial_x(-\Delta)^{-s}$, $0 < s < 1$, is the nonlocal operator:

$$\partial_x \widehat{(-\Delta)^{-s}} u(\xi) = -i\,\xi\,|\xi|^{-2s}\,\widehat{u}(\xi).$$

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Properties:

- Existence and uniqueness of viscosity solutions for (IP)
- Existence (and uniqueness) of weak energy solutions for (P)
- Explicit self-similar solutions, compactly supported for (P)

P.Biler, G.Karch and R.Monneau *Nonlinear Diffusion of Dislocation Density and Self-Similar Solutions*, Commun. Math. Phys. 294, 145–168 (2010).

Higher dimensions

Caffarelli and Vázquez ($N \geq 1$):

$$\partial_t u = \nabla \cdot (u \nabla (-\Delta)^{-s}(u)),$$

where $(-\Delta)^{-s}$ is the Riesz potential, $0 < s < 1$:

$$(-\Delta)^{-s}(u) = K_s \star u = \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^{N-2s}} dy, \quad K_s(x) = C_{N,s}|x|^{-(N-2s)}.$$

$\Longrightarrow \quad \partial_{x_k}(-\Delta)^{-s}$ are the Riesz transforms.

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Difficulties: no comparison principle, nonlinear and nonlocal

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References:

- Existence and finite speed of propagation ($\rightarrow$ free boundaries!) Caffarelli and Vázquez, ARMA 2011.
- Asymptotic behavior: Caffarelli and Vázquez, DCDS 2011.
- Regularity: Caffarelli, Soria and Vázquez, JEMS 2013.
- Exponential convergence towards stationary states in 1D: Carrillo, Huang, Santos and Vázquez, JDE 2015.

General Nonlinearity

$$\partial_t u = \nabla \cdot (u \nabla (-\Delta)^{-s}(u))$$

$\swarrow$ $\searrow$

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-s}(u))$$

Stan, del Teso, Vázquez, 2019.

- Finite/infinite speed of propagation
- Non-explicit self similar solutions
- no comparison principle, no uniqueness

$$\partial_t u = \nabla \cdot (\nabla (-\Delta)^{-s}(u^{n-1}))$$

Biler, Imbert, Karch, 2015.

- Finite speed of propagation (Imbert, Coll.Math. 2016)
- Explicit self similar solutions
- no comparison principle, no uniqueness

Our work

Porous Medium with nonlocal pressure

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla p), \quad p = (-\Delta)^{-s}(u). \qquad \text{(PMFP)}$$

for $x \in \mathbb{R}^N$, $t > 0$, $N \geq 1$. We take $m > 1$, $0 < s < 1$ and $u(x,t) \geq 0$.

The initial data $u(x,0) = u_0(x)$ for $x \in \mathbb{R}^N$, $u_0 : \mathbb{R}^N \to [0, \infty)$ is assumed to be a bounded integrable function.

D. Stan, F. del Teso and J.L. Vázquez, *Finite and infinite speed of propagation for porous medium equations with nonlocal pressure*, JDE 2016.

D. Stan, F. del Teso and J.L. Vázquez, *Existence of weak solutions for a general porous medium equation with nonlocal pressure*, Arch. Rat. Mech. and Analysis 2019.

D. Stan, F. del Teso and J.L. Vázquez, *Porous medium equation with nonlocal pressure*, survey, Springer Volume 2018.

First Approach

Regularize the kernel of $(-\Delta)^{-s}$

$$\mathcal{K}_\epsilon = \rho_\epsilon \star \frac{1}{|x|^{N-2s}}$$

$\Downarrow$

Energy estimates

$\Downarrow$

passing to the limit in the approximation using suitable compactness estimates

Existence for $m \in (1,2)$

Theorem. Let $m \in (1,2)$. Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then there exists a weak solution u of problem (PMFP) with the properties:

- Conservation of mass: For all $t > 0$ we have $\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx$.
- L^∞ estimate: For all $t > 0$ we have $||u(\cdot,t)||_\infty \leq ||u_0||_\infty$.
- Energy estimate: For all $t > 0$

$$C \int_0^t \int_{\mathbb{R}^N} |\nabla(-\Delta)^{-s/2}(u)|^2 dxdt + \int_{\mathbb{R}^N} u(t)^{3-m} dx = \int_{\mathbb{R}^N} u_0^{3-m} dx,$$

with $C = (2-m)(3-m) > 0$.

Existence and Exponential decay in $|x|$ for $m \in [2, 3)$

Theorem. Let $m \in [2, 3)$. Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be s.t.

$$0 \le u_0(x) \le Ae^{-a|x|} \text{ for some } A, a > 0.$$

Then there exists a weak solution u of problem (PMFP) with the properties:

- Conservation of mass, L^∞ estimate.
- Exponential decay: $u(x,t) \le Ae^{Ct-a|x|}$, for $x \in \mathbb{R}^N$, $0 < t \le T$.
- Energy estimate

$$-|C| \int_0^t \int_{\mathbb{R}^N} |\nabla \mathcal{H}(u)|^2 dx dt + \int_{\mathbb{R}^N} u(t)^{3-m} dx = \int_{\mathbb{R}^N} u_0^{3-m} dx,$$

with $C = (2-m)(3-m) < 0$ when $m \in (2, 3)$.

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with $C = (2 - m)(3 - m) < 0$ when $m \in (2, 3)$.

Difficult: the case $m \ge 3$!

New idea: existence for all $m > 1$ when $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$

Based on suitable energy methods.

Formally:

$$\int_{\mathbb{R}^N} u_0^p(x)dx - \int_{\mathbb{R}^N} u(x,t)^p dx = C_1 \int_0^t \int_{\mathbb{R}^N} u^{m+p-2}(-\Delta)^{1-s} u \, dx \, dt$$

$$\geq C_2 \int_0^t \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{1-s}{2}} u^{\frac{m+p-2}{2}} \right|^2 dx dt$$

by the Stroock-Varoupolos Inequality.

Here $C_1 = (p-1)/(m+p-2)$.

New approximation method

$$u_t = \nabla \cdot (u^{m-1} \nabla (-\Delta)^{-1} (-\Delta)^{1-s} u)$$

Then we approximate the operator $\mathcal{L} = (-\Delta)^{1-s}$ by

$$\mathcal{L}_\epsilon^{1-s}(u)(x) = C_{N,1-s} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{(|x-y|^2 + \epsilon^2)^{\frac{N+2-2s}{2}}} dy.$$

- Convergence: $\mathcal{L}_\epsilon^{1-s}[u] \to (-\Delta)^{1-s} u$ pointwise in $\mathbb{R}^N$ as $\epsilon \to 0$
- Generalized Stroock-Varopoulos Inequality for $\mathcal{L}_\epsilon^s$: Let $u \in L^2(\mathbb{R}^N)$. Let $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi \in C^1(\mathbb{R})$ and $\psi' \geq 0$. Then

$$\int_{\mathbb{R}^N} \psi(u) \mathcal{L}_\epsilon^s[u] dx \geq \int_{\mathbb{R}^N} \left| (\mathcal{L}_\epsilon^s)^{\frac{1}{2}} [\Psi(u)] \right|^2 dx,$$

where $\psi' = (\Psi')^2$.

Approximating problem

We consider the approximating problem $(P_{\epsilon\delta\mu R})$

$$\begin{cases} (U_1)_t = \delta\Delta U_1 + \nabla\cdot((U_1+\mu)^{m-1}\nabla(-\Delta)^{-1}\mathcal{L}_\epsilon^{1-s}[U_1]), & (x,t)\in B_R\times(0,T \\ U_1(x,0)=\widehat{u}_0(x), & x\in B_R, \\ U_1(x,t)=0, & x\in\partial B_R,\ t\in(0, \end{cases}$$

with parameters $\epsilon,\delta,\mu,R>0$.

• Existence of solutions of $(P_{\epsilon\delta\mu R})$ $\to$ fixed points of the following map given by the Duhamel's formula

$$\mathcal{T}(v)(x,t) = e^{\delta t\Delta}u_0(x) + \int_0^t \nabla e^{\delta(t-\tau)\Delta}\cdot G(v)(x,\tau)d\tau,$$

where $G(v)=(v+\mu)^{m-1}\nabla(-\Delta)^{-1}\mathcal{L}_\epsilon^s[v]$ and $e^{t\Delta}$ is the Heat Semigroup.

• Existence of solutions of (P)

$$(P_{\epsilon\delta\mu R}) \xrightarrow{\epsilon\to0} (P_{\delta\mu R}) \xrightarrow{R\to\infty} (P_{\delta\mu}) \xrightarrow{\mu\to0} (P_\delta) \xrightarrow{\delta\to0} (P).$$

Existence of weak solutions for $m > 1$

Theorem. Let $1 < m < \infty$, $N \geq 1$, and let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and nonnegative. Then we prove:

- Existence of a weak solution $u \geq 0$ of Problem (P) with data u_0.
- Conservation of mass: For all $0 < t < T$: $\displaystyle\int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx.$
- L^∞ estimate: $||u(\cdot,t)||_\infty \leq ||u_0||_\infty$, $\forall 0 < t < T$
- L^p energy estimate: For all $1 < p < \infty$ and $0 < t < T$ we have

$$\int_{\mathbb{R}^N} u^p(x,t)dx + C(m,p)\int_0^t \int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{1-s}{2}} u^{\frac{m+p-1}{2}}\right|^2 dxdt \leq \int_{\mathbb{R}^N} u_0^p(x)dx.$$

- Second energy estimate: For all $0 < t < T$ we have

$$\frac{1}{2}\int_{\mathbb{R}^N} \left|(-\Delta)^{-\frac{s}{2}} u(t)\right|^2 dx + \int_0^t \int_{\mathbb{R}^N} u^{m-1} \left|\nabla(-\Delta)^{-s} u(t)\right|^2 \leq \frac{1}{2}\int_{\mathbb{R}^N} \left|(-\Delta)^{-\frac{s}{2}} u_0\right|^2 dx.$$

Smoothing effect

Theorem

Let $u \geq 0$ be a constructed weak solution of Problem (P) with $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$. Then

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \leq C_{N,s,m,p}\, t^{-\gamma_p}\, \|u_0\|_{L^p(\mathbb{R}^N)}^{\delta_p} \quad \text{for all} \quad t > 0,$$

where $\gamma_p = \frac{N}{(m-1)N+2p(1-s)}$, $\delta_p = \frac{2p(1-s)}{(m-1)N+2p(1-s)}$.

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where $\gamma_p = \frac{N}{(m-1)N+2p(1-s)}$, $\delta_p = \frac{2p(1-s)}{(m-1)N+2p(1-s)}$.

$\Rightarrow$ Existence of weak solutions for only $u_0 \in L^1(\mathbb{R}^N)$.

$\Rightarrow$ Existence of weak solutions for only $u_0 \in \mathcal{M}^+(\mathbb{R}^N)$.

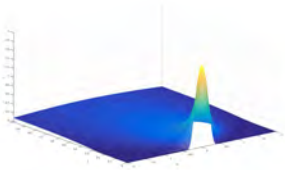

Figure: $m = 1.5,\ s = 0.25$

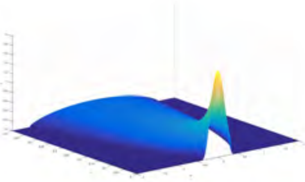

Figure: $m = 2,\ s = 0.25$

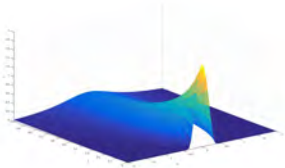

Figure: $m = 1.5,\ s = 0.75$

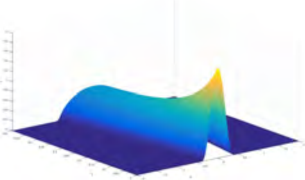

Figure: $m = 2,\ s = 0.75$

$\rightarrow$ Finite speed of propagation for $m \geq 2$ / infinite speed of propagation for $1 < m < 2$

Finite speed of propagation for $m \geq 2$

Theorem

Assume that u_0 has compact support and $u(x,t)$ is bounded for all x, t. Then $u(\cdot, t)$ is compactly supported for all $t > 0$.

If

$$u_0(x) \leq U_0(x) := a(|x| - b)^2,$$

then $\exists\ C = C(t)$ large enough s.t.

$$u(x,t) \leq \mathcal{U}(x,t) := a(Ct - (|x| - b))^2.$$

Consequence: Free Boundaries!

Figure: $u_0 \leq \mathcal{U}_0$

Figure: $u(x,t) \leq \mathcal{U}(x,t)$

Infinite speed of propagation for $m \in (1,2)$ and $N = 1$

Theorem

Let $m \in (1,2)$, $s \in (0,1)$ and $N = 1$. Let u be the solution of Problem (P) with data $u_0 \geq 0$. Then $u(x,t) > 0$ for all $t > 0$, $x \in \mathbb{R}$.

Idea of the proof: Prove that
$v(x,t) = \int_{-\infty}^{x} u(y,t)dy > 0$ for $t > 0$, $x \in \mathbb{R}$.

The integrated problem

$$\partial_t v = -|v_x|^{m-1}(-\Delta)^{1-s} v \quad \text{(II}$$

The initial data is given by
$v_0(x) = \int_{-\infty}^{x} u_0(y)dy$.
$\Longrightarrow \exists$ unique viscosity solution v

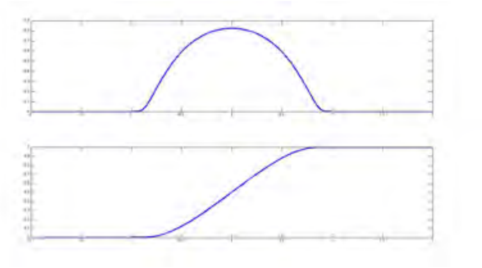

Figure: Typical initial data for models (P) and (IP).

Uniqueness: partial results in 1D

Theorem

Let $m \in (1, +\infty)$*,* $s \in (0, 1)$*,* $N = 1$ *and* $\mu \in \mathcal{M}^+(\mathbb{R}^N)$*. Then there exists a unique constructed weak solution to Problem (P).*

The proof is done via the integrated problem.

Asymptotic behavior

Theorem

Let $m \in (1, \infty)$, $s \in (0,1)$ *and* $N = 1$. *Assume that* $u_0 \in L^1(\mathbb{R})$, $\|u_0\|_{L^1(\mathbb{R})} = M$ *and let* u *be the constructed weak solution of (P). Then*

$$t^{\frac{N(1-\frac{1}{p})}{(m-1)N+2-2s}} \|u(\cdot,t) - U_M(\cdot,t)\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as} \quad t \to \infty$$

for any $p > 1$, *for a unique* U_M *that is a self-similar solution of (P) with initial data* $\mu = M\delta_0$.

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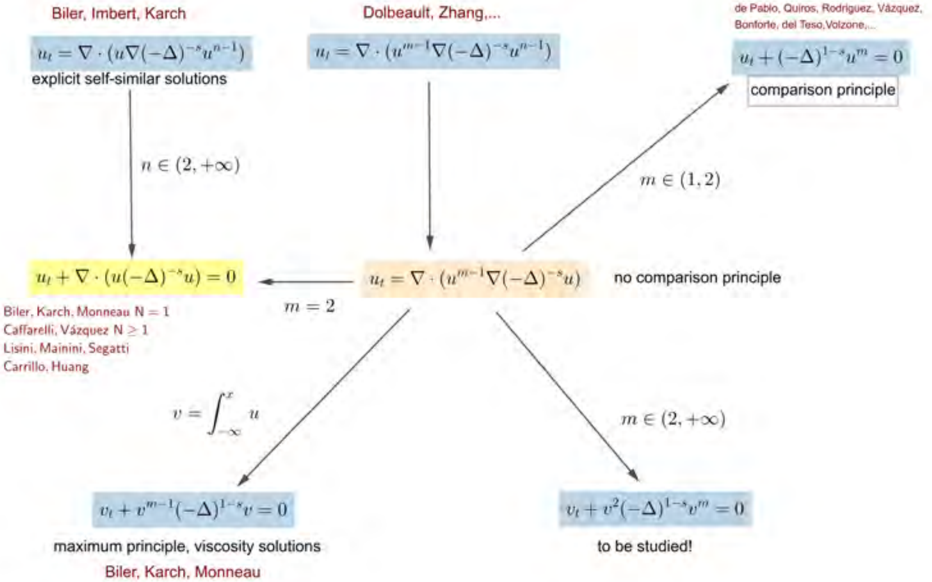
for any $p > 1$, *for a unique* U_M *that is a self-similar solution of (P) with initial data* $\mu = M\delta_0$.

Notice that U_M can be transformed into a self-similar solution of other nonlocal PDEs as we will explain.

Transformation to other nonlocal models



Explicit Solutions

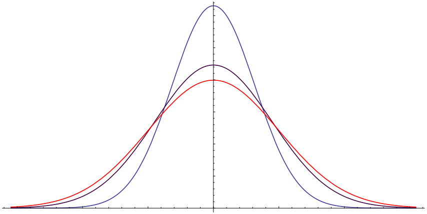
Fractional Porous Medium Equation

$$U_t + (-\Delta)^s U^m = 0 \qquad \text{(FPME)}$$

$\forall s \in (0,1) \rightarrow m = \frac{N+2-2s}{N+2s} > m_c = \frac{N-2s}{N}$

$$u(x,t) = at^{-N\beta_1}\left(R^2 + \left|xt^{-\beta_1}\right|^2\right)^{-(N+2s)/2}$$

Huang, Bul.Lon.Math.Soc. 2014.



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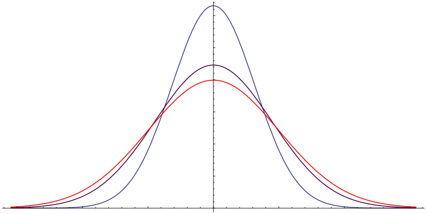
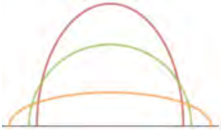


Porous Medium with Fractional Pressure

$$V_t = \nabla \cdot (V\nabla(-\Delta)^{-\tilde{s}} V^{\tilde{m}-1}),\ \tilde{m} > 1$$

Biler, Imbert, Karch [2013]:

$$v(x,t) = at^{-N\beta_2}(R^2 - |xt^{-\beta_2}|^2)^{(1-\tilde{s})/(\tilde{m}-1)}$$



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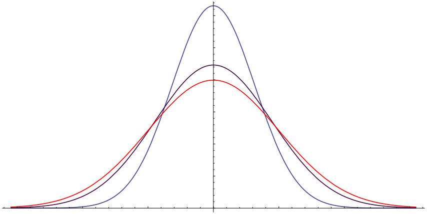
$\forall s \in (0,1) \to m = \frac{N+2-2s}{N+2s} > m_c = \frac{N-2s}{N}$

$$u(x,t) = at^{-N\beta_1}\left(R^2 + \left|xt^{-\beta_1}\right|^2\right)^{-(N+2s)/2}$$

Huang, Bul.Lon.Math.Soc. 2014.



Porous Medium with Fractional Pressure

$$V_t = \nabla \cdot (V \nabla (-\Delta)^{-\tilde{s}} V^{\tilde{m}-1}),\ \tilde{m} > 1$$

Biler, Imbert, Karch [2013]:

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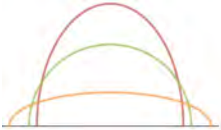

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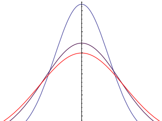

Explicit Solutions

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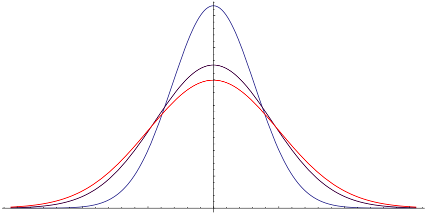

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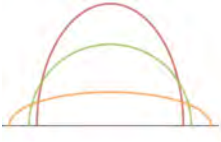

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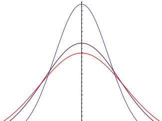

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Thank you for your attention!