The one-sided nature of fractional derivatives

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References

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  "Maximum principles, extension problem and inversion for nonlocal one-sided equations", *J. Differential Equations* **260** (2016), 6333–6362.

- P. R. Stinga and M. Vaughan:
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The usual derivative

For $f = f(t) : \mathbb{R} \to \mathbb{R}$ we define

$$\frac{d}{dt}f(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \qquad \text{whenever } \textit{the limit exists}$$

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The definition involves **two-sided** limits:

$$D_+ f(t) = \lim_{h \to 0^+} \frac{f(t+h) - f(t)}{h}$$ **“from the future / right-sided”**

$$D_- f(t) = \lim_{h \to 0^+} \frac{f(t) - f(t-h)}{h}$$ **“from the past / left-sided”**

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If f is *good enough* then we will have

$$D_+ f(t) = D_- f(t)$$

and this common number is called

$$\frac{d}{dt} f(t)$$

Origin of Fractional Calculus

If $\dfrac{d}{dt} f(t)$ exists, we may try to take higher order derivatives:

$$\frac{d^2}{dt^2} f(t) = \frac{d}{dt}\left(\frac{d}{dt} f\right)(t)$$

and so on:

$$\frac{d^n}{dt^n} f(t) = \underbrace{\frac{d}{dt} \circ \cdots \circ \frac{d}{dt}}_{n-\text{times}} f(t) \qquad \text{(Leibniz notation)}$$

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- Question from L'Hôpital to Leibniz in a letter on **Sep. 30, 1695**:

"What if n be $1/2$?"

- Leibniz's answer:

"It will lead to a paradox. From this apparent paradox, one day useful consequences will be drawn"

B. Ross, A brief history and exposition of the fundamental theory of fractional calculus (1975).

Definitions of fractional derivatives

(1) **Riemann–Liouville left-sided.** $\alpha > 0$, $n - 1 < \alpha < n$, intervals $[a, \infty)$

$$ {}^{RL}D^{\alpha}_{a+} f(t) = \frac{d^n}{dt^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f(h)}{(t-h)^{\alpha-n+1}}\,dh\right] $$

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(4) **Right-sided** versions also available...

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Hint. A defining *algebraic property* for the 1/2-derivative would be

$$\frac{d^{1/2}}{dt^{1/2}}\left(\frac{d^{1/2}}{dt^{1/2}} f\right) = \frac{d}{dt} f$$

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- Is there some way of treating derivatives in an *algebraic* way?

Fourier transform (**1822**)

Fourier transform of $f = f(t)$:

$$\mathcal{F}(f)(\omega) = \widehat{f}(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

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- **Important Property 1.** Inversion Theorem:

$$\mathcal{F}^{-1}(\mathcal{F}(f)) = f$$

Fourier transform and derivatives

Derivatives:

$$\mathcal{F}\left(\frac{d}{dt}f\right)(\omega) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{d}{dt}f(t)e^{-i\omega t}\,dt$$
$$= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}f(t)(i\omega)e^{-i\omega t}\,dt = (i\omega)\mathcal{F}(f)(\omega)$$

or, which is the same,

$$\mathcal{F}^{-1}\big((i\omega)\mathcal{F}(f)\big) = \frac{d}{dt}f$$

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Higher order derivatives:

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- **Important Property 2.** Derivatives correspond to multiplication by $i\omega$:

$$\frac{d^n}{dt^n}f \quad \longleftrightarrow \quad (i\omega)^n\mathcal{F}(f)$$

Fourier transform and fractional derivatives

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It satisfies the defining algebraic property:

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Question. Is there some formula for fractional derivatives like the usual one

$$\frac{d}{dt} f(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \ ?$$

Gamma function (with Bernardis, Martín-Reyes, Torrea)

With the **Cauchy Integral Theorem** we prove that for $i\omega \in i\mathbb{R}$

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- **Important Property 3.** Translations correspond to multiplication by $e^{-h(i\omega)}$:

$$g_h(t) := f(t-h) \quad \longleftrightarrow \quad \mathcal{F}(g) = e^{-h(i\omega)}\mathcal{F}(f)$$

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This is exactly the **left-sided Marchaud fractional derivative** ${}^M D_-^\alpha f(t)$ **!**

Understanding the (Marchaud) fractional derivative

$$(D_-)^\alpha f(t) = c_\alpha \int_0^\infty \frac{f(t) - f(t-h)}{h^{1+\alpha}}\, dh = c_\alpha \int_{-\infty}^t \frac{f(t) - f(h)}{(t-h)^{1+\alpha}}\, dh$$

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Fractional power of

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"from the past"

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- $(D_-)^\alpha$: **Nonlocal operator** *into the past*, not just *infinitesimal*.
- It can be checked from the formula that for f smooth

$$\lim_{\alpha\to 1}(D_-)^\alpha f = D_- f = f' \quad \lim_{\alpha\to 0}(D_-)^\alpha f = f \quad (D_-)^\alpha (D_-)^\beta f = (D_-)^{\alpha+\beta} f$$

The Fundamental Theorem of Calculus is one-sided

Let f be continuous and define, for any $t > a$,

$$F(t) = \int_a^t f(r)\,dr \equiv I_- f(t) \qquad \textbf{“from the past / left-sided”}$$

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The only values we need are “to the left of t”.

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In other words,

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Derivative from the left inverts the **integral from the left**.

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Lebesgue Differentiation Theorem. By **maximal function** estimates, if $f \in L^1_{\mathrm{loc}}$

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The class of **weights** $w(t) > 0$ for which the limit exists a.e. for every $f \in L^p(w)$:

$$\|f\|_{L^p(w)}^p = \int_{-\infty}^{\infty} |f(t)|^p w(t)\,dt < \infty$$

when $1 < p < \infty$, was characterized by E. Sawyer (*Trans. AMS*, 1986).

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Example. Decreasing functions belong to the class A_p^-.

The original Hardy–Littlewood maximal function (1930)

A MAXIMAL THEOREM WITH FUNCTION-THEORETIC APPLICATIONS.

BY

G. H. HARDY and J. E. LITTLEWOOD.

New College Oxford

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Suppose that $\lambda > 0$, that

$$f(z) = f(re^{i\theta})$$

is an analytic function regular for $r \leqq 1$, and that

$$F(\theta) = \operatorname*{Max}_{0 \leqq r \leqq 1} |f(re^{i\theta})|$$

is the maximum of $|f|$ on the radius θ. Is it true that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F^{\lambda}(\theta)\,d\theta \leqq A(\lambda)\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^{\lambda}\,d\theta,$$

where $A(\lambda)$ is a function of λ only? The problem is very interesting in itself, and the theorem suggested may be expected, if it is true, to have important applications to the theory of functions.

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We write

(8. 1) $$A(x,\xi)=A(x,\xi,f)=\frac{1}{x-\xi}\int_{\xi}^{x} f(t)\,dt \quad (0\leqq \xi < x),\quad A(x,x)=f(x).$$

If $f(x)$ is bounded, $A(x,\xi)$ is bounded; in any case it is continuous in ξ except perhaps for $\xi=x$. We define $\Theta(x)$ by

(8. 2) $$\Theta(x)=\Theta(x,f)=\operatorname*{Max}_{0\leqq\xi\leqq x} A(x,\xi)^{1}=\overline{\operatorname*{bound}}_{0\leqq\xi\leqq x} A(x,\xi).$$

Fractional integral: Gamma function (again)

For any $\alpha > 0$ we define

$$(D_-)^{-\alpha} f \quad \longleftrightarrow \quad (i\omega)^{-\alpha} \mathcal{F}(f)$$

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Inverting the Fourier transform we get the **left-sided (Weyl) fractional integral**:

$$(D_-)^{-\alpha} f(t) = c_{-\alpha} \int_0^\infty \frac{f(t-h)}{h^{1-\alpha}}\, dh = c_{-\alpha} \int_{-\infty}^t \frac{f(h)}{(t-h)^{1-\alpha}}\, dh$$

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When f is continuous, $\lim_{\alpha\to 1}(D_-)^{-\alpha} f = I_- f(t)$ and $(D_-)^{\alpha} \circ (D_-)^{-\alpha} f = f$

One-sided Fundamental Theorem of Fractional Calculus

Theorem (with Bernardis, Martín-Reyes and Torrea, 2016)

Let $0 < \alpha < 1$, $1 < p < 1/\alpha$, $1/q = 1/p - \alpha$, $w \in A^-_{p,q}(\mathbb{R})$, $f \in L^p(w^p)$. *Then*

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By this we mean

$$\lim_{\varepsilon \to 0^+} c_\alpha \int_{-\infty}^{t-\varepsilon} \frac{(D_-)^{-\alpha} f(t) - (D_-)^{-\alpha} f(h)}{(t-h)^{1+\alpha}}\, dh = \lim_{\varepsilon \to 0} T_\varepsilon f(t) = f(t)$$

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- **Idea.** Prove the a.e. pointwise maximal operator estimate

$$\sup_{\varepsilon > 0} |T_\varepsilon f(t)| \leq M^- f(t) = \sup_{h > 0} \frac{1}{h} \int_{t-h}^{t} |f(h)|\, dh$$

and apply that $M^- : L^p(w) \to L^p(w)$ if and only if $w \in A^-_p$, $1 < p < \infty$.

Sobolev spaces

- $f \in L^p$ is in the **Sobolev space** $W^{1,p}$ if there exists $g \in L^p$ such that

$$f'(\varphi) = -\int g\varphi'\,dt \qquad \text{for all } \varphi \in \mathcal{S}$$

and in this case we denote $f' = g \in L^p$. Norm: $\|f\|_{W^{1,p}} = (\|f\|_{L^p} + \|f'\|_{L^p})^{1/p}$

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- **Question 2.** One-sided Sobolev spaces?

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Lemma (with M. Vaughan, 2019)

If $\varphi \in \mathcal{S}_-$ then $(D_+)^\alpha \varphi \in C^\infty$ is supported in $(-\infty, A]$ and

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Given any $f \in L^1_{\text{loc}}$ such that $\displaystyle\int_{-\infty}^{A} \frac{|f(t)|}{1 + |t|^{1+\alpha}} \, dt < \infty$ for all $A \in \mathbb{R}$, we define

$$(D_-)^\alpha f(\varphi) = \int f (D_+)^\alpha \varphi \, dt \quad \text{for any } \varphi \in \mathcal{S}_-$$

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New Definition (with M. Vaughan, 2019)

Let $1 < p < \infty$ and $w \in A_p^-$. We say that $f \in L^p(w)$ is in the **one-sided (weighted) Sobolev space** $W^{1,p}(w)$ *if*

$$f' \in L^p(w)$$

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We already know that for smooth f

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Lemma (with M. Vaughan, 2019)

Let $f \in L^p(w)$, $w \in A_p^-$, $1 < p < \infty$. Then, for any $A \in \mathbb{R}$ and $0 \leq \alpha < 1$,

$$\int_{-\infty}^{A} \frac{|f(t)|}{1+|t|^{1+\alpha}}\,dt < \infty.$$

As a consequence, $(D_-)^\alpha f$ is well defined in the distributional sense as

$$(D_-)^\alpha f(\varphi) = \int f (D_+)^\alpha \varphi\,dt \quad \text{for all } \varphi \in \mathcal{S}_-$$

Characterization of one-sided Sobolev spaces

Recall that $W^{1,p}(w)$, $1 < p < \infty$, is the class of functions

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$$\lim_{\alpha \to 1} (D_-)^\alpha f = f' \qquad \lim_{\alpha \to 0} (D_-)^\alpha f = f$$

in the a.e. and $L^p(w)$ senses.

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Recall that $W^{1,p}(w)$, $1 < p < \infty$, is the class of functions

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Let $f \in L^p(w)$, where $w \in A_p^-$, $1 < p < \infty$.

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Then $f \in W^{1,p}(w)$ and $\lim_{\alpha \to 1} (D_-)^\alpha f = f'$ a.e. and in $L^p(w)$.

(3) *Alternatively, suppose that $(D_-)^\alpha f \in L^p(w)$ and that*

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Then $\lim_{\alpha \to 0} (D_-)^\alpha f = f$ a.e. and in $L^p(w)$.

Remarks

- The main ingredient is the following a.e. pointwise maximal estimate we proved:

$$\sup_{0<\alpha<1}\left|c_\alpha\int_{-\infty}^t\frac{f(t)-f(h)}{(t-h)^{1+\alpha}}\,dh\right|\le C\big(M^-(u')(t)+M^-u(t)\big)$$

Here $f\in L^p(w)$ and $C>0$ is independent of f and p.

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  Here $f\in L^p(w)$ and $C>0$ is independent of f and p.

- Even for the unweighted case of $W^{1,p}$ we got new results.

- We also characterized the weighted Sobolev space $W^{2,p}(\mathbb{R}^n\nu)$, $n\geq 1$, where ν is an $A_p(\mathbb{R}^n)$ Muckenhoupt weight (**two-sided** spaces in nature), using fractional Laplacians:

$$(-\Delta)^s u(x) = c_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy$$

Thank you for your attention!