Presentación de líneas de investigación matemática para estudiantes

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Escuela de Ingeniería y Arquitectura (EINA)
Grupo de Investigación: Aplicaciones de Ecuaciones Diferenciales

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Research lines:

1. Numerical approximation of Partial Differential Equations:
   - Singularly perturbed problems
   - Fractional differential equations

     (Prof. Eugene O'Riordan, Dublin City University, Ireland)

     (Prof. Martin Stynes, Beijing Computational Research Center, China)

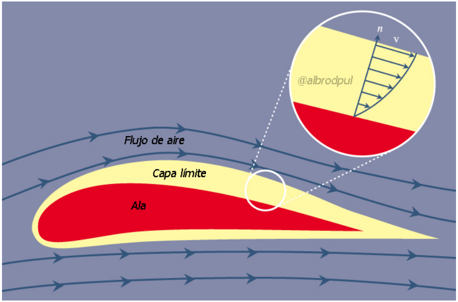
2. Applications:

   (Computational Hydraulics Group, Department of Fluid Dynamics, University of Zaragoza)

Singularly perturbed problems

Some applications: Environment, Fluid dynamics, Quantum mechanics, Elasticity, Chemical reactor theory, Semiconductor devices, Economics models, Physiological models.

Multi-scale character (narrow regions where the solution varies very rapidly, while away from the boundary the solution varies slowly)



Singularly perturbed problems

Reaction–convection–diffusion problems:

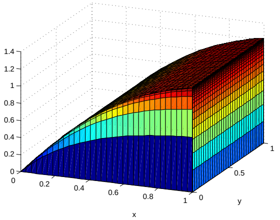
$$\begin{cases} L_\varepsilon u \equiv -\varepsilon \Delta u + \mathbf{a} \cdot \nabla u + bu = f, \quad (x, y) \in \Omega = (0,1)^2, \\ \qquad + \text{ boundary conditions.} \end{cases}$$

- ⋆ $0 < \varepsilon \ll 1$: *Singular perturbation parameter*
- ⋆ $\varepsilon \Delta u$: *Diffusion term*
- ⋆ $\mathbf{a} \cdot \nabla \mathbf{u}$: *Convection term*
- ⋆ bu: *Reaction term*

Singularly perturbed problems

Example:

$$Lu \equiv -\varepsilon \Delta u + (1+x^2)u_x + (1+x)u = 4x(1-x), \; (x,y) \in \Omega = (0,1)^2,$$
$$u(x,y) = 0, \; (x,y) \in \partial\Omega.$$



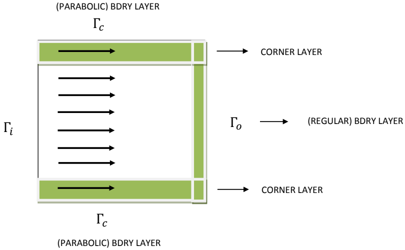


Figure: Computed solution for $\varepsilon = 10^{-6}$

Singularly perturbed problems

Robust or uniformly convergent schemes:

$$\sup_{0<\varepsilon\leq 1} \|U_h - u_h\|_{\infty,d} \leq CN^{-p},$$

where C is a constant independent of N and (also) ε.

Fitted operator methods

(Quasi–)Uniform mesh + special discrete operator

Fitted mesh methods

Layer–adapted mesh + classical discrete operator

From the classical theory one has the crude bounds for the BVP:

$$\left\| \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \right\|_\infty \leq C\varepsilon^{-(i+j)}.$$

Singularly perturbed problems

Singularly perturbed convection-diffusion problem:

$$Lu \equiv -\varepsilon \Delta u + a u_x + bu = f, \quad (x,y) \in \Omega = (0,1)^2, \qquad u \text{ given in } \partial\Omega,$$

where $0 < \varepsilon \leq 1$, $a \geq \alpha > 0$ and $b \geq \beta > 0$.

Decompose the solution into

$$u = v + w_T + w_B + w_R + w_{TR} + w_{BR},$$

where

$$\left| \frac{\partial^{i+j} v}{\partial x^i \partial y^j} \right| \leq C(1 + \varepsilon^{2-(i+j)}), \; 0 \leq i+j \leq 3,$$

$$w_R(x,y) \sim e^{-\alpha(1-x)/\varepsilon}, \quad \partial_x^i w_R(x,y) \sim \varepsilon^{-i} e^{-\alpha(1-x)/\varepsilon},$$

$$w_B(x,y) \sim e^{-\sqrt{\beta/\varepsilon} y}, \quad \partial_y^j w_B(x,y) \sim \varepsilon^{-j/2} e^{-\sqrt{\beta/\varepsilon} y},$$

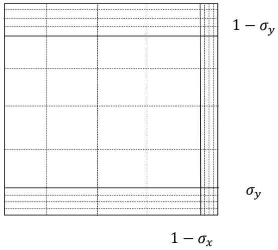
$$w_{TB}(x,y) \sim e^{-\sqrt{\beta/\varepsilon} y} e^{-\alpha(1-x)/\varepsilon}.$$

Singularly perturbed problems

Transition parameters of the Shishkin mesh $\bar{\Omega}^N$

$$\sigma_x = \min\left\{\frac{1}{2}, \frac{\varepsilon}{\alpha}\ln N\right\}$$

$$\sigma_y = \min\left\{\frac{1}{4}, \sqrt{\frac{\varepsilon}{\beta}}\ln N\right\}$$



The mesh is uniform in each one of the subintervals

$$x \text{ variable: } [0, 1-\sigma_x] \cup [1-\sigma_x, 1],$$

$$y \text{ variable: } [0, \sigma_y] \cup [\sigma_y, 1-\sigma_y] \cup [1-\sigma_y, 1].$$

Upwind finite difference scheme

$$L^N U \equiv -\varepsilon\left(\delta^2_{xx} + \delta^2_{yy}\right) U + aD^-_x U + bU = f, \quad \text{in } \Omega^N,$$

where δ^2_{xx}, δ^2_{yy} and D^-_x are the standard central and backward difference approximations.

Singularly perturbed problems

Error analysis

The matrix associated to this scheme is an M-matrix and this scheme is uniformly stable in the maximum norm.
Denote the global e_h and truncation error τ_h by

$$e_h = [u]_h - U_h, \quad L^N e_h = \tau_h.$$

In general, one does not have parameter-uniform bounds of the truncation error τ_h.

but ...

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Result of convergence

The upwind finite difference scheme on the Shishkin mesh satisfies

$$\sup_{0<\varepsilon\leq 1} \|U_h - u_h\|_{\infty,d} \leq CN^{-1} \ln^2 N.$$

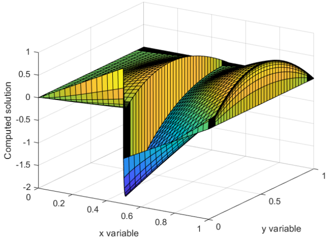
Extensions to other singular perturbed problems

Discontinuous boundary conditions/forcing term

$$-\varepsilon\Delta u + (1+x+y)u_x + (2-xy)u = 64x(1-x)y(1-y),$$
$$(x,y)\in\Omega=(0,1)^2,$$

$$u(x,y)=0,\quad (x,y)\in\partial\Omega\backslash\{y=0\},$$

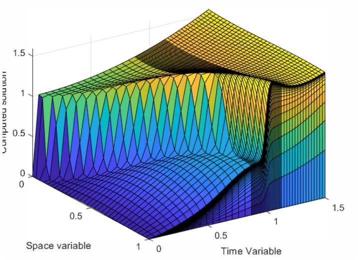
$$u(x,0)=\begin{cases}2x, & \text{if } 0<x<0.5,\\ -2(1-x), & \text{if } 0.5<x<1.\end{cases}$$



Extensions to other singularly perturbed problems

Singularly perturbed initial-boundary value problems with discontinuous data: $u(0^-,t) \neq u(x,0^+)$

$$u_t - \varepsilon u_{xx} + (1+3t^2-2t)u_x = 4x(1-x), \quad (x,t) \in (0,1)\times(0,1.5],$$
$$u(x,0) = x^3(1-x)^3, \quad x \in (0,1),$$
$$u(0,t) = 1 + 0.25t^2, \; u(1,t) = 0, \quad t \in [0,1.5].$$



Fractional differential equations

We consider the following class of problems

$$Lu := D_t^\alpha u - p\frac{\partial^2 u}{\partial x^2} + c(x)u = f(x,t),$$

(+ Initial and boundary conditions)

for $(x,t) \in Q := (0,l) \times (0,T]$ and D_t^α is the Caputo fractional derivative of order α with $0 < \alpha < 1$ and it is defined by

$$D_t^\alpha u(x,t) := \frac{1}{\Gamma(1-\alpha)} \int_{s=0}^{t} \boxed{(t-s)^{-\alpha}} \frac{\partial u}{\partial s}(x,s)\, ds.$$

The definition is *not* local (unlike classical derivatives).

Origins of Fractional Calculus: Letter from L'Hôpital to Leibniz in 1695.

Anomalous dynamics in complex systems and non-local temporal phenomena.

Anomalous diffusion (subdiffusion and superdiffusion processes) explains a number of phenomena in several areas of physics, astrophysics, finance, biology, ecology, geophysics, chemistry, medicine, geology, bioengineering, among others.

Some applications:

- Viscoelastic materials
- Semiconductor materials
- Financial markets
- Transport in plasma
- Control theory
- Porous media (heterogeneous porous aquifer, contaminant transport, seepage flow, fractured reservoirs,...)

Fractional differential equations

The solution in $Q = (0, \pi) \times (0, T]$ of the fractional heat equation

$$D_t^\alpha u = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(\pi, t) = 0, \quad u(x, 0) = \sin(x),$$

is $u(x,t) = E_\alpha(-t^\alpha)\sin(x)$ where

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

(Mittag-Leffler function)

(1846-1927)

and the derivatives of this function satisfy

$$E'_\alpha(-t^\alpha) = \mathcal{O}(t^{\alpha-1}), \quad E''_\alpha(-t^\alpha) = \mathcal{O}(t^{\alpha-2}) \quad \text{as} \quad t \to 0.$$

Therefore, the derivatives of u blow-up at $t = 0$ since $0 < \alpha < 1$.

Fractional differential equations

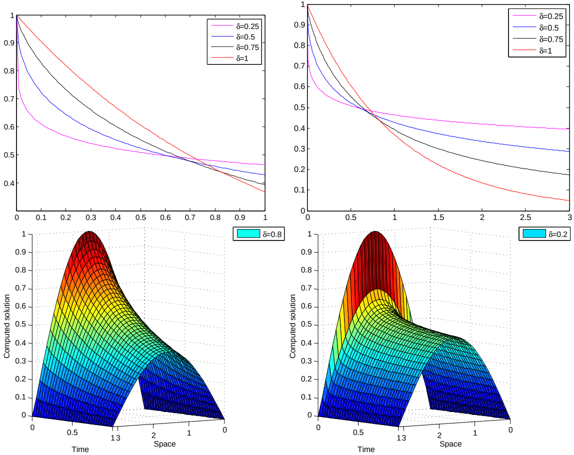


Figure: Mittag-Leffler function and exact solution $u(x,t) = E_\alpha(-t^\alpha)\sin(x)$

Fractional differential equations

Two main difficulties:

i. Non-local operators

ii. Singular behaviour of the solution

Implications:

- Difficulties in the convergence analysis of any numerical method

Fractional differential equations

Two main difficulties:

i. Non-local operators

ii. Singular behaviour of the solution

Implications:

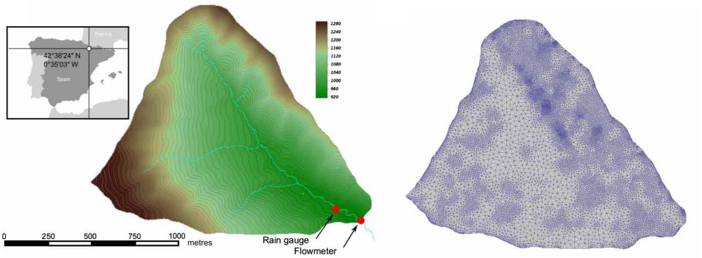
- Difficulties in the convergence analysis of any numerical method

Many open questions:

- Type of fractional derivative (Caputo, Riemann-Liouville, Patie-Simon,. . .)
- Definition of the boundary conditions
- Behaviour of the solution
- Numerical scheme and error analysis, . . .

Applications: Computational Hydraulics Group, UZ

Arnás Catchment



Catchment location and hypsometry map.

Catchment refined mesh.

Figure: Simulation of rainfall/runoff in real catchments

Applications: Computational Hydraulics Group, UZ

The model combines

- 2D shallow water flow model:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} = \mathbf{S} + \mathbf{H} + \mathbf{M}, \quad \mathbf{U} = (h, q_x, q_y)^T.$$

- fractional Green-Ampt infiltration law: it is based on the Darcy's law in the saturated area:

$$q = K_\alpha D_C^\alpha H, \quad 0 < \alpha < 1,$$

where q is the vertical flux, K_α is the hydraulic conductivity and H is the hydraulic head.

Applications: Computational Hydraulics Group, UZ

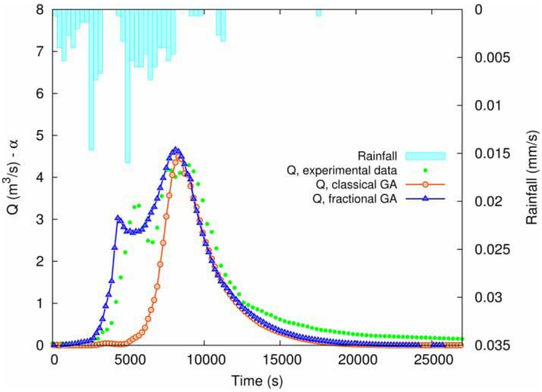


Figure: Hydrograph of outlet discharge Q

[PhD. Thesis (2019)]

Muchas gracias por vuestra atención!!!!

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